# ON A q-ANALOG OF THE MCKAY CORRESPONDENCE AND THE ADE CLASSIFICATION OF $\widehat{\mathfrak{sl}}_2$ CONFORMAL FIELD THEORIES

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ABSTRACT. The goal of this paper is to give a category theory based definition and classification of "finite subgroups in  $U_q(\mathfrak{sl}_2)$ " where  $q=e^{\pi \mathrm{i}/l}$  is a root of unity. We propose a definition of such a subgroup in terms of the category of representations of  $U_q(\mathfrak{sl}_2)$ ; we show that this definition is a natural generalization of the notion of a subgroup in a reductive group, and that it is also related with extensions of the chiral (vertex operator) algebra corresponding to  $\widehat{\mathfrak{sl}}_2$  at level k=l-2. We show that "finite subgroups in  $U_q(\mathfrak{sl}_2)$ " are classified by Dynkin diagrams of types  $A_n, D_{2n}, E_6, E_8$  with Coxeter number equal to l, give a description of this correspondence similar to the classical McKay correspondence, and discuss relation with modular invariants in  $(\widehat{\mathfrak{sl}}_2)_k$  conformal field theory.

The results we get are parallel to those known in the theory of von Neumann subfactors, but our proofs are independent of this theory.

# Introduction

The goal of this paper is to describe a q-analogue of the McKay correspondence. Recall that the usual McKay correspondence is a bijection between finite subgroups  $\Gamma \subset SU(2)$  and affine simply-laced Dynkin diagrams (i.e., affine ADE diagrams). Under this correspondence, the vertices of Dynkin diagram correspond to irreducible representations of  $\Gamma$  and the matrix of tensor product with  $\mathbb{C}^2$  is  $2 - A_{ij}$  where A is the Cartan matrix of the ADE diagram (see [M1]).

There have been numerous results regarding generalization of McKay correspondence with SU(2) replaced by  $U_q(\mathfrak{sl}_2)$  with q being a root of unity,  $q = e^{\pi i/l}$ . Of course, since  $U_q(\mathfrak{sl}_2)$  is not a group, one must first find a reasonable way of making sense of this question. This is usually done by reformulating the problem in terms of representation theory of  $U_q(\mathfrak{sl}_2)$ . More precisely, it is known that the category of finite-dimensional representations of  $U_q(\mathfrak{sl}_2)$  has a semisimple subquotient category  $\mathcal{C}$ , with simple objects  $V_0 \dots, V_k, k = l - 2$  (definition of this category was suggested by Andersen and his collaborators, see [AP]; a review can be found, e.g., in [Fi] or in [BK]). As was shown by Finkelberg [Fi], using results of Kazhdan and Lusztig [KL]', the category  $\mathcal{C}$  is equivalent to the category of integrable  $\widehat{\mathfrak{sl}}_2$ -modules of level k with fusion tensor product. This latter category plays a key role in the Wess–Zumino–Witten model of conformal field theory. Finally, as was shown by Wassermann and his students, fusion in this category can also be described in the language of von Neumann algebras (see, e.g. [W1], [L]).

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Below is an overview of some of the known classification results related to  $U_q(\mathfrak{sl}_2)$ , or, more precisely, to category  $\mathcal{C}$ .

- 1. Ocneanu's classification of subfactors.<sup>1</sup> It is known that to every inclusion of von Neumann factors  $N \subset M$  of finite index one can associate a number of algebraic structures (index, principal graph, relative commutants, etc). In particular, such an inclusion defines a tensor category of N-N bimodules. Ocneanu has suggested that a subfactor  $N \subset M$  with the category of N-N bimodules equivalent to  $\mathcal{C}$  should be considered as a "subgroup of  $U_q(\mathfrak{sl}_2)$ "; thus, classification of subgroups reduces to classification of subfactors. He also gave [O1] a complete classification of such subfactors: they are classified by Dynkin diagrams of types A (which corresponds to trivial inclusion),  $D_{2n}$ ,  $E_6$ ,  $E_8$ , with Coxeter number equal to l. Full proof of this result has been given in the works of Popa [Po], Bion–Nadal [BN1], [BN2], Izumi [I1], [I2].
- 2. Cappelli-Itzykson-Zuber's classification of modular invariants of conformal field theories based on integrable representations of  $\widehat{\mathfrak{sl}}_2$  at level k=l-2 (see [CIZ] or the review in [FMS]). These modular invariants are classified by Dynkin diagrams of ADE type with Coxeter number equal to l. It is known, however, that modular invariants of types  $A, D_{even}, E_6, E_8$  can be obtained from extensions of the corresponding chiral (or vertex operator) algebra, while invariants of the type  $E_7, D_{odd}$  can not be obtained in this way (see [MST]). This classification is related to the previous one: it can be shown that every subfactor  $N \subset M$  gives a modular invariant, see [O2], [O3] and papers of Böckenhauer-Evans-Kawahigashi [BEK, BEK2].
- 3. Etingof and Khovanov's classification of the "integer" modules over the Grothendieck ring ("fusion algebra") of  $\mathcal{C}$  (see [EK]). In this classification, all finite Dynkin diagrams and even diagrams with loops appear.

It should be noted that the classification of "subgroups in  $U_q(\mathfrak{sl}_2)$ " given in the theory of subfactors requires good knowledge of von Neumann algebras and subfactors. It is very different from the ideas in the proof of the classical McKay correspondence.

The main goal of the present paper is to study an alternative definition of a subgroup in  $U_q(\mathfrak{sl}_2)$  which uses nothing but the theory of tensor categories (which one has to use anyway to work in  $\mathcal{C}$ ). Namely, a subgroup in  $U_q(\mathfrak{sl}_2)$  is by definition a commutative associative algebra in  $\mathcal{C}$ , i.e. an object  $A \in \mathcal{C}$  with multiplication morphism  $\mu \colon A \otimes A \to A$  satisfying suitably formulated commutativity, associativity and unit axioms and some mild technical restrictions.<sup>2</sup> We argue that this is the right definition for the following reasons:

- 1. If we replace  $\mathcal{C}$  by a category of representations of a reductive group G, then commutative associative algebras in  $\mathcal{C}$  correspond to subgroups of finite index in G.
- 2. If we replace  $\mathcal{C}$  by a category of representations of some vertex operator algebra  $\mathcal{V}$  (which is good enough so that  $\mathcal{C}$  is a modular tensor category, as it happens for all VOA's appearing in conformal field theory), then associative commutative algebras in  $\mathcal{C}$  (with some minor restrictions) exactly correspond

 $<sup>^1\</sup>mathrm{We}$  would like to thank the referee and M. Müger for explaining to us the status of this classification.

<sup>&</sup>lt;sup>2</sup>While we arrived at this definition independently, we are hardly the first to introduce it. This definition had also been suggested by Wassermann [W2] and Müger (unpublished).

to "extensions"  $\mathcal{V}_e \supset \mathcal{V}$  of this VOA; in other words, in this way we recover the notion of extension of a conformal field theory.

3. Every subfactor  $N \subset M$  defines such an algebra in the category of N-N bimodules.

We show that for any modular category  $\mathcal{C}$  a commutative associative algebra  $A \in \mathcal{C}$  gives rise to two different categories of modules over A. One of these categories, Rep A, comes with two natural functors  $F: \mathcal{C} \to \operatorname{Rep} A, G: \operatorname{Rep} A \to \mathcal{C}$ ; F is a tensor functor, so it defines on Rep A a structure of a module category over  $\mathcal{C}$ . There is also a smaller category  $\operatorname{Rep}^0 A$ ; if A is "rigid", then both  $\operatorname{Rep} A$  and  $\operatorname{Rep}^0 A$  are semisimple and  $\operatorname{Rep}^0 A$  is modular. Both of these categories have appeared in the physical literature in the language of extensions of chiral algebra: in particular,  $\operatorname{Rep}^0 A$  is the category of modules over the extended VOA  $\mathcal{V}_e$ , and  $\operatorname{Rep} A$  is the category of "twisted"  $\mathcal{V}_e$ -modules. These modules appear as possible boundary conditions for extended CFT which preserve  $\mathcal{V}$  (see [FS], [PZ] and references therein); sometimes they are also called "solitonic sectors".

Applying this general setup to  $\mathcal{C}$  being the semisimple part of category of representations of  $U_q(\mathfrak{sl}_2)$ , we see that the fusion algebra of Rep A is a module over the fusion algebra of  $\mathcal{C}$ , which gives a relation with Etingof-Khovanov classification mentioned above. Using their results, we get the following classification theorem which we consider to be the q-analogue of McKay correspondence:

**Theorem.** Commutative associative algebras in C are classified by the (finite) Dynkin diagrams of the types  $A_n, D_{2n}, E_6, E_8$  with Coxeter number equal to l. Under this correspondence, the vertices of the Dynkin diagram correspond to irreducible representations  $X_i \in \text{Rep } A$  and the matrix of tensor product with  $F(V_1)$  in this basis is 2-A, where A is the Cartan matrix of the Dynkin diagram and  $V_1$  is the fundamental (2-dimensional) representation of  $U_q(\mathfrak{sl}_2)$ .

Since  $\operatorname{Rep}^0 A$  is modular, each of these algebras gives a modular invariant providing a relation with the ADE classification of Cappelli-Itzykson-Zuber.

The first part of this theorem — that is, that commutative associative algebras are classified by Dynkin diagrams—is hardly new; in the language of extensions of a chiral algebra, it has been (mostly) known to physicists long ago (see, e.g. [MST]), and these extensions have been studied in a number of papers. However, the second part of the theorem, which explicitly describes a correspondence in a manner parallel to the classical McKay correspondence to the best of our knowledge is new.

The main result of this theorem is parallel to the classification of finite subgroups in  $U_q(\mathfrak{sl}_2)$  as defined in the theory of subfactors. Not being experts in this theory, we are not describing the precise relation here<sup>3</sup>. We just note that our proofs are completely independent and are not based on the subfactor theory, even though some of the methods we use (most notably, the use of conformal embeddings) are parallel to those used in the subfactor theory.

Finally, it should be noted that the problem of finding C algebras A is closely related to the problem of finding all module categories over C (such module categories also play important role in CFT; in physical literature, they are usually described by a certain kind of 6j-symbols, see [PZ]). Indeed, for every C-algebra A

<sup>&</sup>lt;sup>3</sup>We were recently informed by M. Müger that he is currently writing a series of papers, which, among other things, will give a detailed review of this connection.

the category Rep A is a module category over C. It is expected that in the  $U_q(\mathfrak{sl}_2)$  case, all module categories over C are classified by all ADE Dynkin diagrams with Coxeter number equal to l. Theorem above gives construction of module categories of type  $A_m$ ,  $D_{even}$ ,  $E_6$ ,  $E_8$ ; it is easy to show that a module category of type  $D_{2n+1}$  can be constructed from representations of some associative but not commutative C-algebra. We expect the same to hold for the module category of type  $E_7$ .

**Note.** While working on this paper, we were informed by A. Wassermann and H. Wenzl that they have obtained similar results based on the subfactor theory. In fact, many of the results of this paper coincide with those announced by Wassermann in [W2] (except that we do not use unitary structure), even though we arrived at these results completely independently.

**Note.** It was recently pointed out to us that some of the results about algebras in a braided tensor category and modules over them which we prove in Section 1 have been previously found in [Pa].

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# 1. Algebras and modules

Throughout the paper, we denote by  $\mathcal{C}$  a semisimple abelian category over  $\mathbb{C}$  (most of the results are also valid for any base field k of characteristic zero). We denote by I the set of isomorphism classes of irreducible objects in  $\mathcal{C}$  and fix some choice of representative  $V_i$  for every  $i \in I$ . We always assume that the spaces of morphisms are finite-dimensional; since  $\mathbb{C}$  is algebraically closed, this implies that  $\operatorname{Hom}_{\mathcal{C}}(V_i,V_i)=\mathbb{C}$ . We will denote  $\langle X,Y\rangle=\dim\operatorname{Hom}_{\mathcal{C}}(X,Y)$ ; in particular,  $\langle V_i,X\rangle$  is the multiplicity of  $V_i$  in X which shows that this multiplicity is finite.

We assume that  $\mathcal{C}$  is a rigid balanced braided tensor category (see, e.g., [BK] for a review of the theory of braided tensor categories). The commutativity isomorphism will be denoted by  $R_{V,W}$ ; the associativity isomorphism and other canonical identifications such as  $(V \otimes W)^* \simeq W^* \otimes V^*$  will be implicit in our formulas. Additionally, we require that  $\mathbf{1}$  is a simple object in  $\mathcal{C}$ . We will use the symbol 0 to denote the corresponding index in  $I: V_0 = \mathbf{1}$ .

We denote by  $K(\mathcal{C})$  the complexified Grothendieck ring ("fusion algebra") of the category  $\mathcal{C}$ ; this is a commutative associative algebra over  $\mathbb{C}$  with a basis given by classes  $[V_i]$  of simple objects.

- **1.1. Definition.** An associative commutative algebra A in  $\mathcal{C}$  (or  $\mathcal{C}$ -algebra for short) is an object  $A \in \mathcal{C}$  along with morphisms  $\mu \colon A \otimes A \to A$  and  $\iota_A \colon \mathbf{1} \hookrightarrow A$  such that the following conditions hold:
  - 1. (Associativity) Compositions  $\mu \circ (\mu \otimes id), \mu \circ (id \otimes \mu) \colon A^{\otimes 3} \to A$  are equal.
  - 2. (Commutativity) Composition  $\mu \circ R_{AA} : A \otimes A \to A$  is equal to  $\mu$ .
  - 3. (Unit) Composition  $\mu \circ (\iota_A \otimes A)$ :  $A = \mathbf{1} \otimes A \to A$  is equal to  $\mathrm{id}_A$ .
  - 4. (Uniqueness of unit) dim  $\operatorname{Hom}_{\mathcal{C}}(\mathbf{1}, A) = 1$ .

The notion of a C-algebra is not new; it had been used in many papers (for example, in [R]). However, most authors only use algebras in symmetric tensor categories. Algebras in braided categories were studied in [B]; however, the discussion there is limited to the case where algebra A is "transparent", i.e.  $R_{VA}R_{AV} = id$  for every  $V \in C$ . This setting is too restrictive for us. Most of results in [B] generalize to non-transparent case easily, others (mainly, the results regarding distinction between two categories of modules, Rep A and Rep<sup>0</sup> A) require significant work.

Commutative algebras in a braided tensor category are also discussed in [Pa]. Unfortunately, this paper was only brought to our attention after the first version of the current paper appeared in the electronic archive. For this reason and for for reader's convenience, we give here complete proofs of most of the results; still, we would like to point out that most results of this section were first obtained in [Pa] and [B].

We will frequently use graphs to present morphisms in C, as suggested by Reshetikhin and Turaev. We will use the same conventions as in [BK], namely, the morphisms act "from bottom to top". We will use dashed line to represent A and the graphs shown in Figure 1 to represent  $\mu$  and  $\iota_A$ .



FIGURE 1. Morphisms  $\mu$  and  $\iota_A$ 

With this notation, the axioms of a C-algebra can be presented as shown in Figure 2.

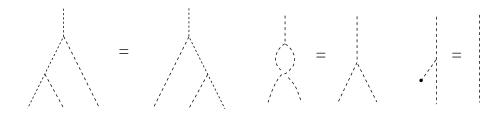


FIGURE 2. Axioms of a commutative associative algebra.

We leave it to the reader to define the notions of morphism of algebras, subalgebras and ideals, quotient algebras etc.

**1.2. Definition.** Let  $\mathcal{C}$  be as above and A — a  $\mathcal{C}$  algebra. Define the category Rep A as follows: objects are pairs  $(V, \mu_V)$  where  $V \in \mathcal{C}$  and  $\mu_V : A \otimes V \to V$  is a morphism in  $\mathcal{C}$  satisfying the following properties:

1. 
$$\mu_V \circ (\mu \otimes id) = \mu_V \circ (id \otimes \mu_V) : A \otimes A \otimes V \to V$$

2. 
$$\mu_V(\iota_A \otimes \mathrm{id}) = \mathrm{id} \colon \mathbf{1} \otimes V \to V$$

The morphisms are defined by

(1.1)  $\operatorname{Hom}_{\operatorname{Rep} A}((V, \mu_V), (W, \mu_W))$ 

$$= \{ \varphi \in \operatorname{Hom}_{\mathcal{C}}(V, W) \mid \mu_{W} \circ (\operatorname{id} \otimes \varphi) = \varphi \circ \mu_{V} \colon A \otimes V \to W \}$$

(see Figure 3).

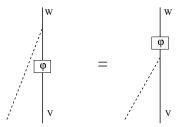


Figure 3. Definition of morphisms in Rep A

An instructive example of such a situation is when G is a finite group and C is the category of finite-dimensional complex representations of G. In this case we will show that semisimple C-algebras correspond to subgroups in G (see Section 2).

1.3. Remark. Contrary to the usual intuition, typically the larger A, the smaller is its category of representations. In the above mentioned example  $\mathcal{C}=\operatorname{Rep} G$ , correspondence between subgroups  $H\subset G$  and  $\mathcal{C}$ -algebras is given by A=F(G/H), so large A corresponds to small H and thus, to small  $\operatorname{Rep} A=\operatorname{Rep} H$ .

Let us study basic properties of Rep A. For brevity, we will use notation  $\operatorname{Hom}_A$  instead of  $\operatorname{Hom}_{\operatorname{Rep} A}$ .

# 1.4. Lemma.

- 1. Rep A is an abelian category with finite-dimensional spaces of morphisms; every object in Rep A has finite length.
- 2.  $\operatorname{Hom}_A(A, A) = \mathbb{C}$ .

*Proof.* Since  $\mathcal{C}$  is an abelian category, it suffices to prove that for  $f \in \operatorname{Hom}_A(V, W)$ , Im f and ker f are actually A-submodules in W, V respectively. The check is straightforward and is left to the reader.

Let  $\varphi \in \operatorname{Hom}_A(A, A)$ . By definition we have:

$$\varphi = \varphi \mu(\mathrm{id}_A \otimes \iota_A) = \mu(\mathrm{id} \otimes \varphi \iota_A)$$

But since **1** has multiplicity one in A, one has  $\varphi \iota_A = c \iota_A$  for some constant c. Thus,  $\varphi = c \mu(\mathrm{id}_A \otimes \iota_A) = c \mathrm{id}$ .

**1.5. Theorem.** Rep A is a monoidal category with unit object A.

*Proof.* Let  $V, W \in \text{Rep } A$ . Define  $V \otimes_A W = V \otimes W / \text{Im}(\mu_1 - \mu_2)$  where  $\mu_1, \mu_2 \colon A \otimes V \otimes W \to V \otimes W$  are defined by

$$\mu_1 = \mu_V \otimes \mathrm{id}_W,$$
  
$$\mu_2 = (\mathrm{id}_V \otimes \mu_W) R_{AV}.$$

This defines  $V \otimes_A W$  as an object of  $\mathcal{C}$ . Define  $\mu_{V \otimes_A W}$  to be  $\mu_1$  or  $\mu_2$  which obviously give the same morphism. One easily sees that this defines a structure of A-module on  $V \otimes W$  and that so defined tensor product is associative. To check that A is the unit object, consider morphisms  $\mu_V \colon A \otimes_A V \to V$  and  $\iota_A \otimes \mathrm{id}_V \colon V \to A \otimes_A V$ . Straightforward check shows that they are well defined, commute with the action of A (that is, satisfy (1.1)) and thus define morphisms in Rep A and finally, that they are inverse to each other.

- **1.6. Theorem.** Define functors  $F: \mathcal{C} \to \operatorname{Rep} A, G: \operatorname{Rep} A \to \mathcal{C}$  by  $F(V) = A \otimes V, \mu_{F(V)} = \mu \otimes \operatorname{id}$  and  $G(V, \mu_{V}) = V$ . Then
  - 1. Both F and G are exact and injective on morphisms.
  - $2.\ F$  and G are adjoint: one has canonical functorial isomorphisms

$$\operatorname{Hom}_A(F(V), X) = \operatorname{Hom}_{\mathcal{C}}(V, G(X)), \qquad V \in \mathcal{C}, X \in \operatorname{Rep} A.$$

- 3. F is a tensor functor: one has canonical isomorphisms  $F(V \otimes W) = F(V) \otimes_A F(W)$ ,  $F(\mathbf{1}) = A$ .
- 4. One has canonical isomorphisms  $G(F(V)) = A \otimes V$  and, more generally,  $G(F(V) \otimes_A X) = V \otimes G(X)$ .

*Proof.* Part (1) is obvious; for part (2), define maps  $\operatorname{Hom}_A(F(V), X) \to \operatorname{Hom}_{\mathcal{C}}(V, G(X))$  and  $\operatorname{Hom}_{\mathcal{C}}(V, G(X)) \to \operatorname{Hom}_A(F(V), X)$  as shown in Figure 4; it is easy to deduce from the axioms that these maps are inverse to each other.

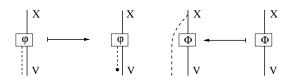


FIGURE 4. Identifications  $\operatorname{Hom}_A(F(V), X) = \operatorname{Hom}_{\mathcal{C}}(V, G(X))$ . Here  $\varphi \in \operatorname{Hom}_A(F(V), X)$ ,  $\Phi \in \operatorname{Hom}_{\mathcal{C}}(V, G(X))$ .

To prove that F is a tensor functor, define functorial morphisms  $f \colon F(V \otimes W) \to F(V) \otimes_A F(W), g \colon F(V) \otimes_A F(W) \to F(V \otimes W)$  by

$$f = \mathrm{id}_A \otimes \mathrm{id}_V \otimes \iota_A \otimes \mathrm{id}_W \colon A \otimes V \otimes W \to (A \otimes V) \otimes_A (A \otimes W)$$

$$g \colon (A \otimes V) \otimes_A (A \otimes W) \xrightarrow{R_{AV}^{-1}} A \otimes_A A \otimes V \otimes V \xrightarrow{\mu} A \otimes V \otimes W.$$

It is immediate to check that they are well-defined and inverse to each other.  $\Box$ 

**1.7. Corollary.** Rep A is a module category over C, i.e. there is an additive functor  $\boxtimes : C \times \text{Rep } A \to \text{Rep } A$  and isomorphisms

$$(V_1 \otimes V_2) \boxtimes X \simeq V_1 \boxtimes (V_2 \boxtimes X), \qquad V_1, V_2 \in \mathcal{C}, X \in \operatorname{Rep} A$$
  
 $\mathbf{1} \boxtimes X \simeq X, \quad X \in \operatorname{Rep} A$ 

satisfying usual compatibility conditions.

*Proof.* Suffices to take 
$$V \boxtimes X = F(V) \otimes_A X$$
.

In particular, this implies that the Grothendieck group K(Rep A) is a module over the Grothendieck ring  $K(\mathcal{C})$ .

However, it is not true that  $\operatorname{Rep} A$  is a braided tensor category. In order to get a braided structure, we need to consider a smaller category.

- **1.8. Definition.** Rep<sup>0</sup> A is the full subcategory in Rep A consisting of objects  $(V, \mu_V)$  such  $\mu_V \circ R_{VA}R_{AV} = \mu_V$ .
- 1.9. Remark. In [Pa], such A-modules are called "dyslectic".

If  $\mathcal{C}$  is symmetric, then  $\operatorname{Rep}^0 A = \operatorname{Rep} A$ . More generally, the same holds if A is "transparent", or "central", in  $\mathcal{C}$  (that is,  $R_{VA}R_{AV} = \operatorname{id}$  for every  $V \in \mathcal{C}$ ); this is the situation considered in [B]. However, in many interesting cases A is not central, and  $\operatorname{Rep} A \neq \operatorname{Rep}^0 A$ .

Later we will justify this definition by showing that if  $\mathcal{C}$  is a category of representation of some vertex operator algebra, and A is an extended vertex operator algebra, then  $\operatorname{Rep}^0 A$  (and not  $\operatorname{Rep} A!$ ) is exactly the category of representations of the vertex operator algebra A.

**1.10. Theorem** ([Pa]). The category  $\operatorname{Rep}^0 A$  is a braided tensor category, with the commutativity isomorphism inherited from  $\mathcal{C}$ .

*Proof.* Let us first show that for  $X,Y \in \operatorname{Rep}^0 A$ ,  $X \otimes_A Y \in \operatorname{Rep}^0 A$ . This follows from the sequence of identities shown in Figure 5. The notation  $f_1 \equiv f_2$  for  $f_1, f_2 \colon A \otimes X \otimes Y \to X \otimes Y$  means that the induced operators  $A \otimes (X \otimes Y) \to X \otimes_A Y$  are equal, i.e.  $p \circ f_1 = p \circ f_2$ , where  $p \colon X \otimes Y \to X \otimes_A Y$  is the canonical projection.

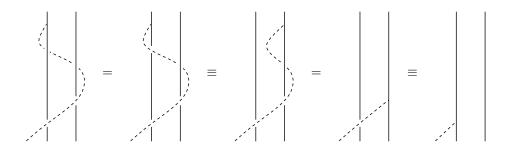


Figure 5.

Next, we need to show that the commutativity isomorphism  $R_{XY} \colon X \otimes Y \to Y \otimes X$  descends to isomorphism  $X \otimes_A Y \to Y \otimes_A X$ . This is equivalent to showing that  $R_{XY}(I) \subset I$ , where  $I = \text{Im}(\mu_1 - \mu_2)$  is the kernel of the canonical projection

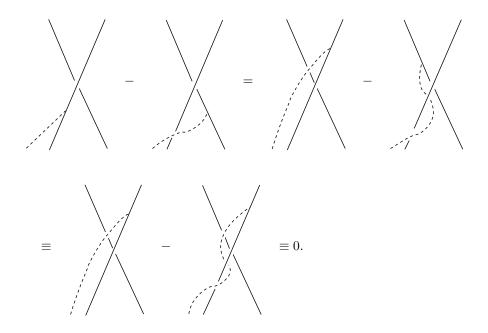


Figure 6.

 $X \otimes Y \to X \otimes_A Y$ . To do so, let us rewrite composition  $R_{XY} \circ (\mu_1 - \mu_2)$  as shown in Figure 6. Thus,  $R_{XY}(\mu_1 - \mu_2) \equiv 0$ , or, equivalently,  $R_{XY}(I) \subset I$ .

Abusing the language, we will also use the same notation  $R_{X,Y}$  for the descended morphisms  $X \otimes_A Y \to Y \otimes_A X$ . Then it is immediate from the definition that these morphisms are A-morphisms. Finally, since the commutativity isomorphism in  $\mathcal{C}$  satisfies the hexagon axioms, the same must hold for the descended operators; thus, the descended operators  $R_{X,Y} \colon X \otimes_A Y \to Y \otimes_A X$  define a structure of a braided tensor category on  $\operatorname{Rep}^0 A$ .

Analysis of this proof also shows the reason why the larger category  $\operatorname{Rep} A$  is not braided: the last identity in Figure 6 would fail.

We also need to know whether categories Rep A, Rep<sup>0</sup> A are rigid. Define  $\varepsilon_A \colon A \to \mathbf{1}$  so that  $\varepsilon_A \iota_A = \mathrm{id}$  (recall that  $\langle A, \mathbf{1} \rangle = 1$ , so this condition uniquely defines  $\varepsilon_A$ ). We will use the graph shown in Figure 7 to represent  $\varepsilon_A$ .



FIGURE 7. Morphism  $\varepsilon_A$ 

We will say that a C-morphism  $f: V \otimes W \to \mathbf{1}$  defines a non-degenerate pairing if f defines an isomorphism  $V \simeq W^*$ , i.e. there exists a map  $g: \mathbf{1} \to W \otimes V$  such that f, g satisfy the rigidity axioms.

**1.11. Definition.** A C-algebra A is called rigid if the map

$$(1.2) e_A: A \otimes A \xrightarrow{\mu} A \xrightarrow{\varepsilon_A} \mathbf{1}$$

is a non-degenerate pairing and  $\dim_{\mathcal{C}} A \neq 0$ .

If A is rigid, then there is a unique morphism  $i_A \colon \mathbf{1} \to A \otimes A$  such that  $e_A, i_A$  satisfy the rigidity axioms: the compositions  $A \xrightarrow{\mathrm{id} \otimes i_A} A \otimes A \otimes A \xrightarrow{e_A \otimes \mathrm{id}} A$ ,  $A \xrightarrow{i_A \otimes \mathrm{id}} A \otimes A \otimes A \xrightarrow{\mathrm{id} \otimes e_A} A$  are equal to identity (uniqueness follows from a well-known fact that the dual object is unique up to a unique isomorphism). We will frequently use the following simple lemma.

# **1.12. Lemma.** Both $e_A$ , $i_A$ are symmetric:

$$e_A R_{AA} = e_A,$$
$$R_{AA} i_A = i_A.$$

*Proof.* The identity for  $e_A$  immediately follows from commutativity of multiplication. For  $i_A$ , it suffices to prove that  $R_{AA}i_A$  satisfies the rigidity axioms, i.e. both compositions below are equal to identity

$$A \xrightarrow{\operatorname{id} \otimes R_{AA}i_A} A \otimes A \otimes A \xrightarrow{e_A \otimes \operatorname{id}} A,$$

$$A \xrightarrow{R_{AA}i_A \otimes \operatorname{id}} A \otimes A \otimes A \xrightarrow{\operatorname{id} \otimes e_A} A.$$

To prove the first identity, we represent the composition by a graph and manipulate it as shown in Figure 8. The second identity is proved in the same manner.

$$|\widetilde{l_A}| = |\widetilde{l_A}| = |\mathrm{id}_A|$$

FIGURE 8. Proof of  $R_{AA}i_A = i_A$ .

Finally, we also need to discuss the following subtle point. The map  $e_A$  allows us to identify  $A \simeq A^*$  and thus, we can also identify  $A \simeq A^{**}$ . On the other hand, in any balanced rigid braided category one has a canonical identification  $\delta_V : V \simeq V^{**}$ . It is natural to ask whether these two identifications coincide. The answer is given by the following lemma.

**1.13. Lemma.** Let A be a rigid algebra. Then the morphism  $A \simeq A^{**}$  defined by  $e_A$  coincides with  $\delta_A : A \to A^{**}$  iff  $\theta_A = \mathrm{id}$ .

*Proof.* Recalling the relation between the twists  $\theta_V$  and  $\delta_V$  (see, e.g., [BK, Section 2.2]), we see that the statement of the lemma is equivalent to the following equation:

$$[e_{A}]$$
  $= \theta_{A}.$ 

But it easily follows from symmetry of  $e_A$  that the right hand side is the identity morphism.

This lemma shows that if A is rigid,  $\theta_A = \mathrm{id}$ , then we can identify  $A \simeq A^*$  so that the canonical morphisms  $A \otimes A^* \to \mathbf{1}$ ,  $A^* \otimes A \to \mathbf{1}$  are both given by  $e_A$ , and the morphisms  $\mathbf{1} \to A \otimes A^*, \mathbf{1} \to A^* \otimes A$  are both given by  $i_A$ . As usual, we will use "cap" and "cup" to denote  $e_A$ ,  $i_A$  in the figures. Then the statement of Lemma 1.12 can be graphically presented as follows:

$$(1.3) \qquad \qquad = \qquad$$

We will frequently use the following easy lemma.

**1.14. Lemma.** If A is a rigid C-algebra,  $\theta_A = id$ , then

$$(1.4) \qquad \qquad = \dim A$$

The proof is immediate if we note that both sides are morphisms  $\mathbf{1} \to A$  and by uniqueness of unit axiom must be proportional.

**1.15. Theorem.** If C is a rigid category, A — a rigid C-algebra,  $\theta_A = \mathrm{id}$ , then the categories Rep A, Rep<sup>0</sup> A are rigid.

*Proof.* Let  $(V, \mu_V) \in \text{Rep } A$ . Define the dual object  $(V^*, \mu_{V^*})$  as follows:  $V^*$  is the dual of V in  $\mathcal{C}$  and  $\mu_{V^*}$  is defined by Figure 9.

This definition implies the following identities:

Define now the maps  $\tilde{i}_V \in \operatorname{Hom}_A(A, V \otimes_A V^*), \tilde{e}_V \in \operatorname{Hom}_A(V^* \otimes_A V, A)$  by Figure 11 (we leave it to the reader to check that these formulas indeed define morphisms in Rep A).

It is easy to check by using identities in Figure 10 and isomorphisms  $A \otimes_A V \simeq V$  defined in the proof of Theorem 1.5 that these two maps satisfy the rigidity axioms.

# 1.16. Lemma. Let A be a rigid C-algebra. Then

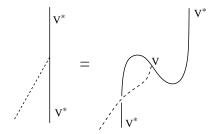


FIGURE 9. Definition of dual object in Rep A.

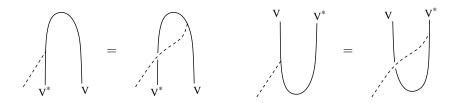


FIGURE 10.

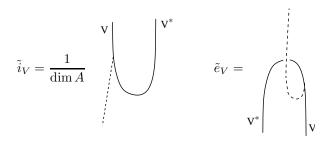


FIGURE 11. Rigidity maps in  $\operatorname{Rep} A$ 

 $1.\ F\ and\ G\ are\ 2-sided\ adjoints\ of\ each\ other:\ in\ addition\ to\ results\ of\ Theorem\ 1.6,\ we\ also\ have\ canonical\ isomorphisms$ 

$$\operatorname{Hom}_A(X, F(V)) = \operatorname{Hom}_{\mathcal{C}}(G(X), V), \qquad V \in \mathcal{C}, X \in \operatorname{Rep} A.$$

2.  $F(V^*) \simeq (F(V))^*$ .

*Proof.* To prove part (1), we construct linear maps between  $\operatorname{Hom}_A(X, F(V))$  and  $\operatorname{Hom}_{\mathcal{C}}(G(X), V)$  as shown in Figure 12; we leave it to the reader to check that these maps are inverse to each other.

To prove (2), note that as object of C,  $(F(V))^* = V^* \otimes A^* = V^* \otimes A$ , where we used rigidity to identify  $A = A^*$ . Consider the morphism  $R_{AV^*} : A \otimes V^* \to V^* \otimes A$ .

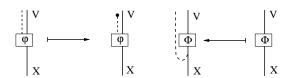


FIGURE 12. Identifications  $\operatorname{Hom}_A(X, F(V)) = \operatorname{Hom}_{\mathcal{C}}(G(X), V)$ . Here  $\varphi \in \operatorname{Hom}_A(X, F(V)), \ \Phi \in \operatorname{Hom}_{\mathcal{C}}(G(X), V)$ .

Again, we leave it to the reader to check that this morphism is actually a morphism of A-modules  $F(V^*) \to (F(V))^*$ .

This shows that Rep A is rigid. To prove rigidity of Rep<sup>0</sup> A, it suffices to show that for  $X \in \text{Rep}^0 A$ ,  $X^* \in \text{Rep}^0 A$  which easily follows from the definition of Rep<sup>0</sup> A and the definition of dual object given by Figure 9.

Finally, we need to check discuss whether Rep A, Rep<sup>0</sup> A are balanced. Recall that balancing in a rigid braided tensor category is a system of functorial isomorphisms  $\delta_V \colon V \simeq V^{**}$  satisfying conditions

(1.5) 
$$\delta_{V \otimes W} = \delta_{V} \otimes \delta_{W},$$
$$\delta_{\mathbf{1}} = \mathrm{id},$$
$$\delta_{V^{*}} = (\delta_{V}^{*})^{-1},$$

where we have used canonical identifications  $(V \otimes W)^* = W^* \otimes V^*$  and for  $F: X \to Y$ ,  $f^*: Y^* \to X^*$  is the adjoint morphism.

This is equivalent to defining a system of functorial morphisms  $\theta_V \colon V \to V$  (twists), satisfying

(1.6) 
$$\begin{aligned} \theta_{V \otimes W} &= R_{WV} R_{VW} \theta_V \otimes \theta_W, \\ \theta_1 &= \mathrm{id}, \\ \theta_{V^*} &= (\theta_V)^*. \end{aligned}$$

(see, for example, [BK, Section 2.2]).

**1.17. Theorem.** Let C be a rigid balanced braided category, and A—a rigid C-algebra,  $\theta_A = \mathrm{id}$ . Then

- 1. Rep<sup>0</sup>  $A = \{V \in \text{Rep } A \mid \theta_V \text{ is an } A\text{-morphism } \}.$
- 2.  $\operatorname{Rep}^0 A$  is a rigid balanced braided category, with  $\theta$  inherited from  $\mathcal{C}$ .
- 3. For any  $V \in \text{Rep } A$ , the morphism  $\delta_V : V \to V^{**}$  is an A-morphism.

*Proof.* Part (1) follows from  $\theta_{A\otimes V}=R_{VA}R_{AV}\theta_{A}\otimes\theta_{V}$  and  $\theta_{A}=\mathrm{id}$ ; (2) immediately follows from (1). To prove (3), it suffices to prove that  $\delta_{V}^{-1}\colon V^{**}\to V$  is an Amorphism. We can rewrite  $\delta_{V}^{-1}$  in terms of  $\theta$  as follows (see [BK, Section 2.2]):

$$\delta_V^{-1} = V V^*$$

$$V^*$$

It now follows from the identities shown in Figure 13 (which uses (1.6) and identities from Figure 10) that  $\delta^{-1}$  is an A-morphism.

FIGURE 13. Proof that  $\delta_V^{-1}$  is an A-morphism

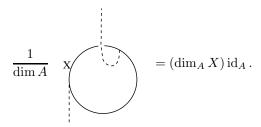
This theorem shows that the category Rep A, which is a rigid monoidal category, while not braided, does have a system of functorial morphisms  $\delta_X \colon X \to X^{**}$  satisfying (1.5). Such categories are sometimes called "pivotal"; in such a category, one can define for every object two numbers, its "left" and "right" dimension (see, e.g., [BW]). We will denote by  $\dim_A X$  the "left" dimension of an object  $X \in \operatorname{Rep} A$ .

**1.18. Theorem.** Let  $\mathcal{C}$  be a rigid balanced braided category, and A-a rigid  $\mathcal{C}$ -algebra such that  $\theta_A=\operatorname{id}_A$ . Then for every  $X,Y\in\operatorname{Rep} A$ ,  $\dim_A(X\otimes_AY)=\dim_A(X)\dim_A(Y)$  and

$$\dim_A(X) = \frac{\dim_{\mathcal{C}}(X)}{\dim_{\mathcal{C}} A},$$
  
$$\dim_A(F(V)) = \dim_{\mathcal{C}}(V).$$

*Proof.* Formula  $\dim_A(X \otimes_A Y) = \dim_A(X) \dim_A(Y)$  holds in any pivotal category and can be easily deduced from (1.5).

Using definition of rigidity morphisms in Rep A shown in Figure 11, we see that  $\dim_A X$  is defined by the following identity:



Both sides are A-morphisms  $A \to A$ . Composing them with  $\iota_A, \varepsilon_A$ , we get  $\frac{1}{\dim A} \dim_{\mathcal{C}} X = \dim_A X$ .

Applying this to  $X = F(V) = V \otimes A$ , we get

$$\dim_A F(V) = \frac{1}{\dim A} \dim_{\mathcal{C}} F(V) = \frac{1}{\dim A} (\dim A) \cdot (\dim V) = \dim V.$$

As a useful corollary, we get the following result:

(1.7) 
$$\dim_{\mathcal{C}}(X \otimes_{A} Y) = \frac{\dim_{\mathcal{C}}(X) \dim_{\mathcal{C}}(Y)}{\dim_{\mathcal{C}}(A)}.$$

1.19. Remark. In the theorem above, we could have used "right" dimension instead of the "left" dimension (this would require minor change in the proof). Thus, we see that both left and right dimension of  $X \in \operatorname{Rep} A$  are equal to  $\frac{\dim_{\mathcal{C}}(X)}{\dim_{\mathcal{C}} A}$ ; in particular, they are equal to each other. In a similar way, one could prove that for each A-morphism  $F \colon X \to X$ , its left and right dimension are equal; in terminology of [BW],  $\operatorname{Rep} A$  is a spherical category.

For future use, we note the following somewhat unusual result.

**1.20. Lemma.** Let C be rigid and A - a C-algebra such that  $\theta_A = \mathrm{id}$ ,  $\dim_{\mathcal{C}} A \neq 0$ . Then A is a rigid C-algebra iff A is simple as an A-module.

*Proof.* Let A be rigid; assume  $I \subset A$  is a submodule. By rigidity,  $\mathbf{1} \subset \mu(A \otimes I)$ . On the other hand, since I is a submodule, this implies that  $\mathbf{1} \subset I$ . By unit axiom, this implies I = A.

Conversely, assume that A is simple as A-module. Consider  $A^* \in \mathcal{C}$  and define on it the action of A as in Theorem 1.15. Then one easily sees that the morphism

$$A \xrightarrow{\mathrm{id} \otimes i_A} A \otimes A \otimes A^* \xrightarrow{e_A \otimes \mathrm{id}} A^*,$$

where  $e_A$  is as in (1.2) is a morphism of A-modules. On the other hand, usual arguments show that if A is a simple A-module, then so is  $A^*$ . Thus, such a map is either zero (impossible because of the unit axiom) or an isomorphism.

Finally, recall that we defined  $X \otimes_A Y$  as a quotient of  $X \otimes Y$ . It turns out that in the rigid case,  $X \otimes_A Y$  can also be described as a sub-object of  $X \otimes Y$  and thus, as a direct summand.

**1.21. Lemma.** If A is a rigid C-algebra,  $X, Y \in \text{Rep } A \text{ and } Q \colon X \otimes Y \to X \otimes Y$  is as shown in Figure 14, then  $Q^2 = Q$  and  $\ker Q = \ker(X \otimes Y \to X \otimes_A Y)$ .

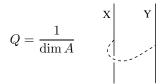


FIGURE 14. Projector on  $X \otimes_A Y \subset X \otimes Y$ 

Proof of this lemma is left to the reader as an exercise.

**1.22.** Corollary. If A is a rigid C-algebra, then one has a canonical direct sum decomposition  $X \otimes Y = Z \oplus X \otimes_A Y$  for some  $Z \in \mathcal{C}$ .

Indeed, it suffices to take  $Z = \ker Q$  and identify  $X \otimes_A Y = \operatorname{Im} Q$ .

# 2. Example: groups and subgroups

In this section we discuss an important example of the general setup discussed above. Namely, let G be a group such that the category  $\mathcal{C}$  of finite-dimensional complex representations of G is semisimple (for example, G is a finite group or G is a reductive Lie group). Also, for a finite set X let F(X) be the space of complex-valued functions on X.

**2.1. Theorem.** If  $H \subset G$  is a subgroup of finite index, then the space A = F(G/H) of functions on G/H is a semisimple C-algebra and Rep A is equivalent to the category Rep H of representations of H; under this equivalence the functors F and G are identified with the restriction and induction functor respectively:

$$F = \operatorname{Res}_H^G \colon \operatorname{Rep} G \to \operatorname{Rep} H,$$
$$G = \operatorname{Ind}_H^G \colon \operatorname{Rep} H \to \operatorname{Rep} G$$

*Proof.* By definition, an object of Rep A is a G-module V with a decomposition  $V=\oplus_{x\in G/H}V_x$  such that  $gV_x=V_{gx}$ , and tensor product in Rep A is given by  $(V\otimes_A W)_x=V_x\otimes W_x$ . Define functor Rep  $A\to \operatorname{Rep} H$  by  $\oplus V_x\mapsto V_1$  and Rep  $H\to \operatorname{Rep} A$  by  $E\mapsto \operatorname{Ind}_H^G E$  (note that it follows from definition of the induced module that  $V=\operatorname{Ind} E$  has a natural decomposition  $V=\oplus_{x\in G/H}V_x$ ). It is trivial to check that these functors preserve tensor product and are inverse to each other.

**2.2. Theorem.** For C = Rep G, any rigid C-algebra is of the form F(G/H) for some subgroup G of finite index.

*Proof.* First, a C-algebra is just a commutative associative algebra over  $\mathbb C$  on which G acts by automorphisms. Next, if A is rigid, then A is semisimple as a commutative associative algebra over  $\mathbb C$ . Indeed, let N be the radical of A; then N is invariant under the action of G and thus is an ideal in A in the sense of C-algebras. By Lemma 1.20, N=0.

Thus, A is the algebra of functions on a finite set X (which can be described as the set of primitive idempotents of A) and G acts by permutations on X. Since  $\mathbb{C}$  appears in decomposition of A as G-module with multiplicity one, this implies that the action of G on X is transitive, so X = G/H.

# 3. Semisimplicity

As before, we let  $\mathcal{C}$  be a braided tensor category.

**3.1. Definition.** A C-algebra is called semisimple if Rep A is semisimple.

We will be mostly interested in the case when  $\mathcal{C}$  is rigid and balanced. In this case, semisimplicity of Rep A implies semisimplicity of Rep A.

- **3.2. Theorem.** Let C be rigid balanced, and A a semisimple C-algebra with  $\theta_A = \mathrm{id}$ . Then
  - 1. If  $X \in \text{Rep } A \text{ is simple, then } X \in \text{Rep}^0 A \text{ iff } \theta_X = c \cdot \text{id.}$
  - 2. Rep<sup>0</sup> A is semisimple, with simple objects  $X_{\pi}$  where  $X_{\pi}$  is a simple object in Rep A such that  $\theta_X = c \cdot id$ .

*Proof.* Immediately follows from Theorem 1.17 and the fact that for a simple object  $X \in \text{Rep } A$ ,  $\text{Hom}_A(X,X) = \mathbb{C}$ .

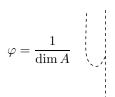
The main result of this section is the following theorem.

**3.3. Theorem.** Let C be rigid, and A — a rigid C-algebra. Then A is semisimple.

*Proof.* The proof is based on the following lemma.

**3.4. Lemma** ([B]). If A is rigid, then every  $X \in \text{Rep } A$  is a direct summand in F(V) for some  $V \in \mathcal{C}$ .

*Proof of the lemma*. Consider the map  $\mu$ :  $A \otimes A \to A$ . It is surjective and is a morphism of A-modules. Moreover, both A and  $A \otimes A$  have canonical structures of A-bimodules, and  $\mu$  is a morphism of A-bimodules (we leave the definition of A-bimodule as an exercise to the reader). This map has one-sided inverse: if we define  $\varphi$ :  $A \to A \otimes A$  by



then  $\varphi$  is a morphism of A-bimodules and it immediately follows from Lemma 1.14 that  $\mu\varphi = \mathrm{id}_A$ . Thus,  $A \otimes A$  splits: we can write

$$A \otimes A \simeq A \oplus Z$$

for some A-bimodule Z so that under this isomorphism,  $\mu$  is the projection on the first summand.

Therefore, 
$$A \otimes_A X \simeq X$$
 is a direct summand of  $(A \otimes A) \otimes_A X = A \otimes (A \otimes_A X) = A \otimes X = F(G(X))$ .

From this lemma, the proof is easy. Indeed, it easily follows from exactness of G and adjointness of F and G (see Theorem 1.6) that for every  $V \in \mathcal{C}$ , F(V) is a projective object in Rep A. Since a direct summand of a projective object is projective, the lemma implies that every  $X \in \text{Rep } A$  is projective and thus,  $\text{Ext}^1(X,Y) = 0$  for every  $X,Y \in \text{Rep } A$ .

3.5. Remark. Morally, this theorem is parallel to the following well known result in Lie algebra theory: if the Killing form on  $\mathfrak g$  is non-degenerate, then the category of finite-dimensional representations of  $\mathfrak g$  is semisimple (this is combination of Cartan's criterion of semisimplicity and Weyl's complete reducibility theorem). The proof, of course, is completely different.

# 4. Modularity

Recall that a semisimple balanced rigid braided category  $\mathcal C$  is called modular if it has finitely many isomorphism classes of simple objects  $V_i, i \in I, |I| < \infty$  and the matrix  $\tilde{s}_{ij}$  defined by Figure 15 is non-degenerate. (We will also use numbers  $\tilde{s}_{VW}$  defined in the same way as  $\tilde{s}_{ij}$  but with  $V_i$  replaced by  $V, V_j$  replaced by W).

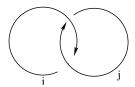


FIGURE 15. Matrix  $\tilde{s}_{ij}$ 

In this case, it is known that the matrices

(4.1) 
$$s_{ij} = \frac{1}{D}\tilde{s}_{ij},$$
 
$$t_{ij} = \frac{1}{\zeta}\delta_{ij}\theta_{i},$$

where  $D, \zeta$  are some non-zero numbers, satisfy the relations of  $SL_2(\mathbb{Z})$ :  $(st)^3 = s^2, s^4 = \text{id}$ . These matrices are naturally interpreted as matrices of some operators s, t acting on  $K(\mathcal{C})$ ; for example, the operator  $s = \frac{1}{D}\tilde{s}$  where

$$\tilde{s}[V] = \sum \tilde{s}_{VV_j}[V_j]$$

where [V] is the class in K of  $V \in \mathcal{C}$ .

We also note that the numbers D,  $\zeta$  appearing in (4.1) are determined uniquely up to a simultaneous change of sign. The number  $D=(s_{00})^{-1}$  is sometimes called the rank of  $\mathcal{C}$ . If  $\mathcal{C}$  is Hermitian category, it is possible to choose D to be a positive real number. In modular tensor categories coming from conformal field theory, the number  $\zeta$  is given by  $\zeta=e^{2\pi i c/24}$ , where c is the (Virasoro) central charge of the theory.

In this section we assume that  $\mathcal{C}$  is a modular tensor category and A is a rigid  $\mathcal{C}$ -algebra, which satisfies  $\theta_A = \operatorname{id}$ ; by Theorem 3.3, this implies that A is semisimple. We denote isomorphism classes of simple objects in Rep A by  $X_{\pi}, \pi \in \Pi$ , and let K(A) be the fusion algebra of Rep A. Similarly, set of simple objects in Rep<sup>0</sup> A is  $\Pi^0 \subset \Pi$  (see Theorem 3.2) and the fusion algebra of Rep<sup>0</sup> A is  $K^0(A) \subset K(A)$ . We will denote by  $P: K(A) \to K^0(A)$  the projection operator:  $P([X_{\pi}]) = [X_{\pi}]$  if  $\pi \in \Pi^0$  and  $P([X_{\pi}]) = 0$  otherwise.

Define operator  $\tilde{s}^A \colon K^0(A) \to K^0(A)$  in the same way as for  $\mathcal{C}$  but replacing  $V_j$  by  $X_{\pi}$  and using rigidity morphisms in Rep<sup>0</sup> A rather than in  $\mathcal{C}$ .

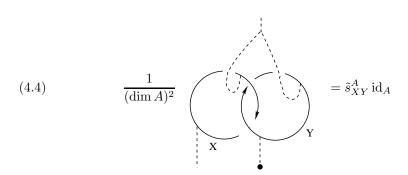
**4.1. Theorem.** Let  $G: K^0(A) \to K$  be the map induced by the functor G from Theorem 1.6, and let  $F^0: K \to K^0(A)$  be the composition PF, where  $P: K(A) \to K^0(A)$  is the projection operator defined above. Then  $G, F^0$  commute with the action of  $\tilde{s}, \tilde{t}$  up to a constant:

$$\begin{split} G(\dim A)\tilde{s}^A &= \tilde{s}G, \qquad F^0\tilde{s} = (\dim A)\tilde{s}^A F^0 \\ G\tilde{t}^A &= \tilde{t}G, \qquad F^0\tilde{t} = \tilde{t}^A F^0. \end{split}$$

To prove this theorem, we will need several technical lemmas.

**4.2.** Lemma. For  $X, Y \in \operatorname{Rep}^0 A$ , the number  $\tilde{s}_{XY}^A$  is given by

*Proof.* Recalling the definition of rigidity isomorphisms in Rep A and isomorphisms  $A \otimes_A A \xrightarrow{\sim} A$ , we see that  $\tilde{s}_{XY}^A$  is given by



Restricting both sides to  $\mathbf{1} \subset A$ , we get

$$\tilde{s}_{XY}^A = \frac{1}{(\dim A)^2}$$

$$X$$

which is easily seen to be equivalent to the statement of the lemma.

**4.3. Lemma.** Let  $X_{\pi} \in \text{Rep } A$  be simple. Define  $P_{\pi} : X_{\pi} \to X_{\pi}$  by

$$(4.6) P_{\pi} = \frac{1}{\dim A} = \frac{1}{\dim A} X_{\pi}$$

Then

(4.7) 
$$P_{\pi} = \begin{cases} \operatorname{id}_{X_{\pi}}, & \text{if } X_{\pi} \in \operatorname{Rep}^{0} A \\ 0, & \text{otherwise} \end{cases}$$

*Proof.* If  $X_{\pi} \in \operatorname{Rep}^0$ , then the statement immediately follows from Lemma 1.14. Thus, let us assume that  $X_{\pi} \notin \operatorname{Rep}^0 A$  and prove that in this case,  $P_{\pi} = 0$ . First, note that the composition  $\theta_{\pi}^{-1}P_{\pi}$  can be rewritten as shown in Figure 16.

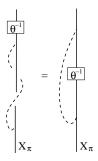


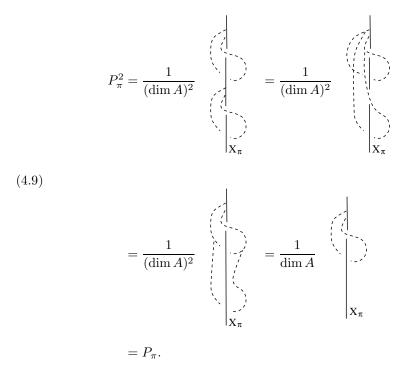
FIGURE 16. Presentation of  $\theta_{\pi}^{-1}P_{\pi}$ 

From this presentation one easily sees that  $\theta_{\pi}^{-1}P_{\pi}$  is a morphism of A-modules; since  $X_{\pi}$  is simple, this implies

$$\theta_{\pi}^{-1} P_{\pi} = c_{\pi} \operatorname{id}$$

for some  $c_{\pi} \in \mathbb{C}$ .

Next, let us calculate  $P_{\pi}^2$ :



Thus,  $P_{\pi}$  is a projector. On the other hand, it follows from (4.8) that  $P_{\pi} = c_{\pi}\theta_{\pi}$ . Combining these two results, we get  $c_{\pi}^2\theta_{\pi} = c_{\pi}$ . If we assume that  $c_{\pi} \neq 0$ , then this implies that  $\theta_{\pi} = c_{\pi}^{-1}$ ; by Theorem 3.2, this is impossible if  $X_{\pi} \notin \text{Rep}^0 A$ . Thus,  $c_{\pi} = 0$ .

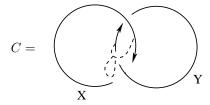
# **4.4. Lemma.** For $X \in \operatorname{Rep}^0 A$ , $Y \in \operatorname{Rep} A$ , one has

$$(4.10) \qquad \langle \tilde{s}^A(X), Y \rangle = \langle \tilde{s}^A(P(Y)), X \rangle = \frac{1}{(\dim A)^2}$$

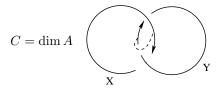
where  $P: K(A) \to K^0(A)$  is as in Theorem 4.1.

*Proof.* Since both sides are linear in Y it suffices to prove this formula when Y is simple. If  $Y \in \text{Rep}^0 A$ , the statement immediately follows from Lemma 4.2. Thus, we only need to prove that if Y is simple,  $Y \notin \text{Rep}^0 A$ , then the right-hand side is

zero. To prove this, let  $C \in \mathbb{C}$  be defined by



On one hand, it easily follows from Lemma 1.14 that



On the other hand, we can deform the figure defining C as shown below

By Lemma 4.3, this implies C = 0.

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Proof for  $\tilde{t}$  is obvious from the definition. As for  $\tilde{s}$ , it suffices to prove that  $(\dim A)\langle G\tilde{s}^A(X),V\rangle=\langle \tilde{s}(G(X)),V\rangle$  for any  $X\in \operatorname{Rep}^0A,V\in\mathcal{C}$ . Using adjointness of G and F, this reduces to

$$(\dim A)\langle \tilde{s}^A(X), F(V)\rangle = \tilde{s}_{G(X),V}.$$

(note that  $F(V) \in \text{Rep } A$ , but in general, not in  $\text{Rep}^0 A$ ). Using Lemma 4.4 and definition of F(V), this can be rewritten as the following identity of figures:

$$(4.12) \qquad \frac{1}{\dim A} \qquad \qquad \bigvee_{\mathbf{X}} \qquad \qquad \bigvee_{\mathbf{V}} \qquad \qquad \bigvee_{\mathbf{V}} \qquad \qquad \bigvee_{\mathbf{V}} \qquad \bigvee_{\mathbf{V} \qquad \bigvee_{\mathbf{V}} \qquad \bigvee_{\mathbf{V}} \qquad \bigvee_{\mathbf{V}} \qquad \bigvee_{\mathbf{V}} \qquad \bigvee_{\mathbf{V}} \qquad \bigvee_{\mathbf{V} \qquad \bigvee_{\mathbf{V}} \qquad \bigvee_{\mathbf$$

which can be proved by rewriting the graph in left hand side as shown in Figure 17 and using Lemma 4.3.

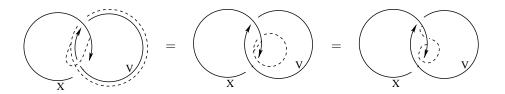


Figure 17.

Similarly, the identity involving  $F^0$  is equivalent to

$$\langle (\dim A)\tilde{s}^A(F^0(V)), X \rangle = \langle \tilde{s}(V), G(X) \rangle$$

which is also equivalent to (4.12). This completes the proof of Theorem 4.1.  $\square$ 

This theorem implies the following important result.

**4.5. Theorem.** If C is a modular category, A is a rigid C-algebra,  $\theta_A = \mathrm{id}$ , then  $\mathrm{Rep}^0 A$  is modular and the numbers  $D, \zeta$  appearing in (4.1) for  $\mathrm{Rep}^0 A$  are related with the corresponding numbers for C by

(4.13) 
$$D(\operatorname{Rep}^{0} A) = \frac{D(\mathcal{C})}{\dim A}$$
$$\zeta(\operatorname{Rep}^{0} A) = \zeta(\mathcal{C}).$$

Also, the maps  $G: K(\operatorname{Rep}^0 A) \to K(\mathcal{C}), F^0: K(\mathcal{C}) \to K(\operatorname{Rep}^0 A)$  commute with operators s,t.

*Proof.* The proof is based on the following lemma.

**4.6. Lemma.** Let A be a semisimple rigid braided balanced category over  $\mathbb{C}$ , with finitely many isomorphism classes of simple objects. Then A is modular iff the matrix  $\tilde{s}$ , defined by Figure 15, satisfies

$$\tilde{s}^2 \mathbf{1} = c \mathbf{1}$$

for some  $c \in \mathbb{C}, c \neq 0$ .

This lemma is not new; however, for the sake of completeness, we include its proof below.

Thus, to prove that  $\operatorname{Rep}^0 A$  is modular, it suffices to prove  $(\tilde{s}^A)^2 A = cA$  for some  $c \neq 0$ . But by Theorem 4.1,  $\tilde{s}^A$  commutes with  $F^0$  up to a constant; thus,

$$(\tilde{s}^A)^2 A = (\tilde{s}^A)^2 F^0(\mathbf{1}) = \frac{1}{(\dim A)^2} F^0(\tilde{s}^2 \mathbf{1}) = \frac{1}{(\dim A)^2} F^0(c\mathbf{1})$$
$$= \frac{c}{(\dim A)^2} A.$$

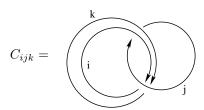
Thus,  $\operatorname{Rep}^0 A$  is modular; all other statements of the theorem immediately follow from Theorem 4.1.

Proof of the lemma. If  $\mathcal{A}$  is modular, the statement is well known and in fact  $c=D^2$  (see, e.g., [BK]). Thus, let us assume that (4.14) holds and deduce from it non-degeneracy of  $\tilde{s}$ .

First, note that  $\tilde{s}\mathbf{1} = \mathbf{d} = \sum d_i V_i$ , where  $V_i$  are simple objects in  $\mathcal{A}$  and  $d_i = \dim_{\mathcal{A}} V_i$ . Thus, (4.14) implies  $\langle \tilde{s}\mathbf{d}, V_i \rangle = c\delta_{i,0}$  which can be rewritten as

$$(4.15) \qquad \sum_{j} d_{j} \qquad \boxed{i} \qquad = c\delta_{i,0} \qquad i$$

Let us now choose some  $i, k \in I$  and let



On one hand, it is easy to show using the definition of  $\tilde{s}$  that

$$C_{ijk} = \frac{\tilde{s}_{ij}\tilde{s}_{jk}}{d_i}$$

(see, e.g., [BK, Lemma 3.1.4]) and thus,

$$\sum_{j} d_j C_{ijk} = \sum_{j} \tilde{s}_{ij} \tilde{s}_{jk} = (\tilde{s}^2)_{ik}.$$

On the other hand, decomposing  $V_i^* \otimes V_k^*$  in a direct sum of irreducibles and using (4.15), we get

$$\sum_{i} d_{j} C_{ijk} = c \langle V_{i}^{*} \otimes V_{k}^{*}, \mathbf{1} \rangle = c \delta_{ik^{*}}$$

which is a non-singular matrix. Therefore,  $\tilde{s}^2$  is non-singular and thus  $\mathcal{A}$  is modular. This completes the proof of the lemma and thus of Theorem 4.5.

4.7. Remark. For modular tensor categories coming from conformal field theory, the identity  $\zeta(\mathcal{C}) = \zeta(\text{Rep}^0 A)$  can be interpreted as stating that an extended CFT has the same central charge as the original CFT, which, of course, should be expected.

#### 5. Vertex operator algebras

In this section, we give the example which was one of our main motivations for this work. Detailed proofs of the results given here will appear in a separate paper [HKL]; here we only outline the main ideas. We should also note that relation between extensions of vertex operator algebras and algebras in a category discussed here was independently found by A. Wassermann [W2], who discussed his work with the second author (Ostrik) during his visit to MSRI in November 2000. At this time, we were finishing the first draft of this paper.

We assume that the reader is familiar with the notion of a vertex operator algebra (VOA); a review and list of references can be found, e.g., in [Fr], [FHL]. To avoid ambiguity, we mention that we include the Virasoro element  $\omega$  and  $\mathbb{Z}$ -grading in the definition of a VOA. Similarly, when talking about modules over a VOA, we always assume that  $L_0$  acts semisimply with finite-dimensional eigenspaces (this automatically gives  $\mathbb{C}$ -grading on a module). We only consider finite length modules.

Let  $\mathcal{V}$  be a vertex operator algebra which is nice enough so that the following properties are satisfied:

- 1. For every simple  $\mathcal{V}$ -module M, its conformal dimension (i.e. lowest eigenvalue of  $L_0$ ) is real,  $\geq 0$ , with equality only for  $M = \mathcal{V}$ , in which case dim  $\mathcal{V}_0 = 1$ .
- 2. The category of representations of  $\mathcal{V}$  is semisimple, with only finitely many simple objects, and all spaces of conformal blocks (i.e., intertwining operators between tensor products of representations) are finite-dimensional. Also,  $\mathcal{V}$  is simple as a  $\mathcal{V}$ -module.
- 3. The category  $\mathcal{C}$  of  $\mathcal{V}$ -modules is a rigid braided tensor category.

The first condition is technical; we will only need it to ensure uniqueness of vacuum vector (see proof of Theorem 5.2 below). The most important condition is the last one, which deserves detailed discussion.

There are at least two ways to define the tensor product (usually called the fusion tensor product) structure on the category of  $\mathcal{V}$ -modules, both originating in the pioneering work of Moore and Seiberg. The first construction, developed in a series of papers of Huang and Lepowsky [HL1], [HL2], [H], is based on defining the tensor product via the space of intertwining operators. The second approach uses the vector spaces of coinvariants (see [Fr]) which should give a modular functor, and then using this modular functor to define the structure of a tensor category (see [BK]). This shows that for every VOA appearing in conformal field theory the category of modules has a structure of a rigid braided tensor category. In fact, such a VOA has to satisfy a stronger restriction:

(3') The category of  $\mathcal{V}$ -modules is a modular tensor category.

Indeed, it follows from the axioms of a rational conformal field theory that the spaces of conformal blocks for such a VOA form a modular functor, and it is known that a modular functor allows one to define a structure of a modular tensor category (see, e.g., [BK]).

These two approaches should give equivalent results; unfortunately, to the best of our knowledge, details of this equivalence are not available in the literature. In

what follows, we will use the first approach, i.e. use the definition of the tensor structure given by Huang and Lepowsky.

In both approaches, it is relatively easy to give the definition of the tensor product, but it is extremely difficult to check that for a given VOA this tensor product is well-defined and defines a structure of a rigid balanced braided category (see [H] for the list of conditions that need to be checked). So far, this has only been checked in very few examples.<sup>4</sup>

Most important example of a VOA for which conditions (1)–(3) have been checked is the VOA coming from an affine Lie algebra at positive integer level, discussed below.

**5.1. Example.** Let  $\mathfrak{g}$  be a simple Lie algebra,  $\widehat{\mathfrak{g}}$  — corresponding affine Lie algebra, and k — a non-negative integer (level). Let  $L_{0,k}$  be the integrable  $\widehat{\mathfrak{g}}$  module of level k with highest weight 0 (the vacuum module). Then it is known that it has a canonical structure of a VOA; we will denote this VOA by  $\mathcal{V}(\mathfrak{g},k)$ . This VOA satisfies requirements (1)–(3') and thus, its category of representations  $\mathcal{C}(\mathfrak{g},k)$  is modular (see [HL3], [BK]). As an abelian category,  $\mathcal{C}(\mathfrak{g},k)$  is just the category of integrable  $\widehat{\mathfrak{g}}$  modules of level k. It is also known (see [Fi]) that  $\mathcal{C}(\mathfrak{g},k)$  is equivalent (as modular category) to the "semisimple part" of the category of representations of the quantum group  $U_q\mathfrak{g}$  with  $q=e^{\pi \mathrm{i}/m(k+h^\vee)}$ , where  $h^\vee$  is the dual Coxeter number and m=1 for simply-laced algebras, m=2 for  $B_n, C_n, F_4$  and m=3 for  $G_2$ .

Let  $\mathcal{V} \subset \mathcal{V}_e$  be a subalgebra (in the sense of VOA's). Assume in addition that  $\mathcal{V}_e$  is finite length as a module over  $\mathcal{V}$ . Then we will call  $\mathcal{V}_e$  an extension of  $\mathcal{V}$ .

- **5.2. Theorem.** Let V be a VOA satisfying (1)–(3) above, and let C be the category of V-modules. Then the following two notions are equivalent:
  - 1. An extension  $V \subset V_e$ , where  $V_e$  is also a VOA satisfying properties (1)–(3) above
  - 2. A rigid C-algebra A with  $\theta_A = 1$

Under this correspondence, category of  $\mathcal{V}_e$ -modules is identified with Rep<sup>0</sup> A.

 ${\it Proof.}$  We give a sketch of the proof; details will appear in the forthcoming paper [HKL].

If  $\mathcal{V}_e$  is a VOA, then for every  $v \in \mathcal{V}_e$  we have the vertex operator  $Y(v,z) \colon \mathcal{V}_e \to \mathcal{V}_e[[z,z^{-1}]]$ . Restricting it to  $v \in \mathcal{V}$ , we get a structure of a  $\mathcal{V}$ -module on  $\mathcal{V}_e$ . It is immediate from the definitions that the map  $Y(\cdot,z)$  is an intertwining operator of the type  $\binom{\mathcal{V}_e}{\mathcal{V}_e}$  and thus gives a morphism of  $\mathcal{V}$ -modules  $Y \colon \mathcal{V}_e \otimes \mathcal{V}_e \to \mathcal{V}_e$ , where  $\otimes$  is the "fusion" tensor product. It follows from the usual commutativity and associativity axioms for a VOA (see [FHL]) that Y defines a structure of a commutative and associative algebra on  $\mathcal{V}_e$ . Existence and uniqueness of unit follow from existence and uniqueness of the vacuum vector in a VOA (see condition (1) above). Condition  $\theta_A = 1$  follows from the fact that eigenvalues of  $L_0$  on  $\mathcal{V}_e$  are integer. A straightforward check shows that the arguments above can be reversed and that the category of representations of  $\mathcal{V}_e$  as a VOA coincides with Rep<sup>0</sup> A.  $\square$ 

<sup>&</sup>lt;sup>4</sup>Of course, as mentioned above, this should hold for any VOA that comes from a rational conformal field theory, but this does not help much: axioms of RCFT are even more difficult to check.

One of the general ways to construct extensions of the VOA  $\mathcal{V}(\mathfrak{g},k)$  is by using the notion of conformal embedding (note, however, that not all extensions can be obtained in this way). Let  $\mathfrak{g} \subset \mathfrak{g}'$  be an embedding of Lie algebras; then it defines an embedding of affine Lie algebras  $\widehat{\mathfrak{g}} \subset \widehat{\mathfrak{g}}'$ . This embedding doesn't preserve the level — a pullback of a  $\widehat{\mathfrak{g}}'$  module of level k' will be a module of level  $k = x_e k'$  for some integer  $x_e$ ; we will symbolically write  $(\widehat{\mathfrak{g}})_k \subset (\widehat{\mathfrak{g}}')_{k'}$ . It defines an embedding  $\mathcal{V}(\mathfrak{g},k) \subset \mathcal{V}(\mathfrak{g}',k')$  which preserves the operator product expansion (i.e., the algebra structure in  $\mathcal{V}$ ) but in general not the Virasoro element. In some special cases, however, such an embedding preserves the Virasoro element as well and therefore defines an embedding of VOA's; they are called *conformal embeddings*. In this case it is easy to show (see, e.g., [FMS, Chapter 17]) that  $\mathcal{V}(\mathfrak{g}',k')$  is automatically finite as  $\widehat{\mathfrak{g}}$ -module, so  $\mathcal{V}(\mathfrak{g}',k')$  is an extension of  $\mathcal{V}(\mathfrak{g},k)$ .

**5.3. Example.** Let  $C(\mathfrak{sl}_2, k)$  be the category of integrable modules over  $\widehat{\mathfrak{sl}}_2$  of level k. Then it is known that for k = 10, there is a conformal embedding  $(\widehat{\mathfrak{sl}}_2)_{10} \subset \widehat{\mathfrak{sp}(4)}_1$ . The easiest way to describe this embedding is to note that the irreducible 4-dimensional representation of  $\mathfrak{sl}_2$  has an invariant non-degenerate skew-symmetric form, which gives an embedding  $\mathfrak{sl}_2 \subset \mathfrak{sp}(4)$ .

The decomposition of  $\mathcal{V}(sp(4),1)$  as  $\mathcal{V}(\mathfrak{sl}_2,10)$  module is given by  $\mathcal{V}=L_{0,10}\oplus L_{6,10}$  (see [FMS, Chapter 17]). Thus, this shows that the object  $A=L_{0,10}\oplus L_{6,10}\in \mathcal{C}(\mathfrak{sl}_2,10)$  has a structure of a rigid  $\mathcal{C}$ -algebra (later we will show that such a structure is unique).

Similarly, for k=28 there exists a conformal embedding  $(\widehat{\mathfrak{sl}}_2)_{28} \subset (\widehat{G}_2)_1$ ; the decomposition of  $\mathcal{V}(G_2,1)$  as  $\mathcal{V}(\mathfrak{sl}_2,28)$  module is given by  $\mathcal{V}=L_{0,28}\oplus L_{10,28}\oplus L_{18,28}\oplus L_{28,28}$ .

- $5.4.\ Remark$ . The use of conformal embeddings to produce extensions of chiral algebras is, of course, well known in physics literature. Conformal embeddings can also be used in the subfactor theory see [X] and references therein.
- 5.5. Remark. It is very easy to prove rigorously that the conformal embedding  $\widehat{\mathfrak{g}}_k \subset \widehat{\mathfrak{g}}'_{k'}$  determines a structure of a rigid  $\mathcal{C}$ -algebra on the vacuum module  $V = L_{0,k'}$  over  $\widehat{\mathfrak{g}}'$  considered as a module over  $\widehat{\mathfrak{g}}$ , even without referring to the more general Theorem 5.2. Let  $\otimes_{\mathfrak{g}}$ ,  $\otimes_{\mathfrak{g}'}$  denote the fusion product over  $\widehat{\mathfrak{g}}$ ,  $\widehat{\mathfrak{g}}'$  respectively. this fusion tensor product can be defined in terms of coinvariants for the action of the algebra of rational  $\mathfrak{g}$  (respectively,  $\mathfrak{g}'$ ) valued functions. Namely, consider the rational curve  $\mathbb{P}^1$  with 3 marked points and the representations  $V, V, V^*$ assigned to these points. The spaces of homomorphisms  $V \otimes_{\mathfrak{g}} V \to V, V \otimes \mathfrak{g}'$ are isomorphic to the dual of the spaces of coinvariants for  $\mathfrak{g}, \mathfrak{g}'$ , see [KL, Fi]. Also, the space of coinvariants for  $\mathfrak{g}'$  is canonically isomorphic to  $\mathcal{C}$ . By definition, we have a surjection  $Coinv(\mathfrak{g}) \rightarrow Coinv(\mathfrak{g}')$ . Thus, we have an embedding  $\operatorname{Hom}(V \otimes_{\mathfrak{g}'} V, V) \subset \operatorname{Hom}(V \otimes_{\mathfrak{g}} V, V)$ . Thus, the canonical morphism  $V \otimes_{\mathfrak{g}'} V \to V$  defines a morphism  $m: V \otimes_h V \to V$ . Let us prove that this morphism defines an associative multiplication, that is  $m(m \otimes_{\mathfrak{q}} id) = m(id \otimes_{\mathfrak{q}} m)$ . Both sides of this equality are represented by some coinvariants (for  $\mathbb{P}^1$  with 4 marked points and the representations  $V, V, V, V^*$  assigned to these points) and by the construction of the fusion product these coinvariants actually come from the coinvariants over g. But the space of the coinvariants over  $\mathfrak{g}'$  is one dimensional since  $V \otimes_{\mathfrak{g}'} V \otimes_{\mathfrak{g}'} V \simeq V$ and hence the LHS and the RHS are proportional. To compute the proportionality

coefficient it is enough to note that m restricts nontrivially on  $V_0 \subset V$  where  $V_0$  is the vacuum module over  $\widehat{\mathfrak{g}}$ . The proof of commutativity is completely analogous. Finally one can use the coinvariant above to identify V and  $V^*$  as  $\widehat{\mathfrak{g}}$ —modules what implies that  $m: V \otimes_h V \to V \to V_0$  can be used to identify V and  $V^*$  as  $\widehat{\mathfrak{g}}$ —modules that is V is rigid  $\mathcal{C}$ —algebra over  $\widehat{\mathfrak{g}}$ .

# 6. ADE CLASSIFICATION FOR $U_q(\mathfrak{sl}_2)$

In this section, we apply the general formalism developed above in a special case: when  $\mathcal C$  is the semisimple part of the category of representations of  $U_q(\mathfrak {sl}_2)$  with  $q=e^{\pi \mathrm{i}/l}, l\leq 2$  as defined by Andersen et al [AP]. We assume that the reader is familiar with the definition and main properties of categories of representations of quantum groups at roots of unity; if not, we refer to the monograph [BK] for a review.

It is known that the category C is semisimple, with simple objects  $V_0, \ldots, V_k$ , where k = l - 2 and  $V_i$  is the usual (i + 1)-dimensional irreducible representation of  $U_q(\mathfrak{sl}_2)$ . Its Grothendieck ring K is generated by one element,  $V_1$ . The quantum dimensions are given by

$$\dim_{\mathcal{C}} V_n = [n+1] := \frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}}$$

which in particular implies that for any non-zero object V,

$$\dim_{\mathcal{C}} V > 1.$$

It is also known that this category is modular and that the universal twist  $\theta$  is given by

(6.2) 
$$\theta_n := \theta_{V_n} = q^{n(n+2)/2} = e^{2\pi i \frac{n(n+2)}{4(k+2)}}.$$

Finally, we note that this category is equivalent to the category of integrable representations of affine Lie algebra  $\widehat{\mathfrak{sl}_2}$  of level k=l-2, or, equivalently, the category of representations of the corresponding vertex operator algebra  $\mathcal{V}(\mathfrak{sl}_2,k)$  (see [Fi]).

Our main goal is to classify all C-algebras.

**6.1. Theorem.** There is a correspondence between rigid C-algebras with  $\theta_A = \mathrm{id}$  and Dynkin diagrams of types  $A_n, D_{2n}, E_6, E_8$  with Coxeter number equal to l. Under this correspondence, simple objects of Rep A correspond to vertices of the Dynkin diagram, and the matrix of multiplication by  $F(V_1)$  in  $K(\operatorname{Rep} A)$  is 2 - A, where A is the Cartan matrix of the Dynkin diagram.

*Proof.* Let A be a rigid C-algebra with  $\theta_A = \mathrm{id}$ . In this case, Rep A is a monoidal category and, by Corollary 1.7, a module category over C. This implies that the Grothendieck ring  $K(A) = K(\operatorname{Rep} A)$  is a module over K(C). By Theorem 3.3, Rep A is semisimple, so K(A) has a distinguished basis (classes  $[X_{\pi}]$  of simple objects) so that in this basis, multiplication by any  $F(V), V \in C$ , has coefficients from  $\mathbb{Z}_+$ . In addition, this module has the following properties:

(i) The module K(A) is indecomposable: it is impossible to split the set of simple objects  $\Pi$  as  $\Pi = \Pi' \sqcup \Pi''$  so that  $K' = \bigoplus_{\Pi'} \mathbb{C}[X_{\pi}], K'' = \bigoplus_{\Pi''} \mathbb{C}[X_{\pi}]$  are  $K(\mathcal{C})$ -submodules in K(A).

Indeed, every simple module  $X_{\pi}$  appears with non-zero multiplicity in  $F(V_i) \otimes_A A = F(V_i)$  for some  $V_i$ . This follows from  $\langle F(V_i), X_{\pi} \rangle = \langle V_i, G(X_{\pi}) \rangle$ .

- (ii) There exists a map  $d: K(A) \to \mathbb{C}$  such that  $d(X_{\pi}) \in \mathbb{R}_{>0}$  and  $d(F(V) \otimes_A X) = (\dim_{\mathcal{C}} V) d(X)$ .
  - Indeed, it suffices to let  $d(X) = \dim_{\text{Rep }A}(X)$  and use Theorem 1.18.
- (iii) There exists a symmetric bilinear form  $\langle , \rangle$  on K(A) such that  $\langle F(V) \otimes_A X, Y \rangle = \langle X, F(V) \otimes_A Y \rangle$  for any  $V \in \mathcal{C}, X, Y \in \operatorname{Rep} A$ . Indeed, we can let  $\langle X, Y \rangle = \dim \operatorname{Hom}_A(X, Y)$  and use rigidity and  $F(V_i)^* \simeq$

 $F(V_i^*) \simeq F(V_i)$  (not canonically).

All modules M over  $K(\mathcal{C})$  which have properties (i)–(iii) above were classified in [EK], where it is shown that they correspond to finite Dynkin diagrams with loops with Coxeter number equal to l. Under this correspondence, vertices of the Dynkin diagram correspond to the elements of distinguished basis of M, and the matrix of multiplication by  $V_1 \in \mathcal{C}$  is 2-A, where A is the Cartan matrix of the Dynkin diagram

(Dynkin diagrams with loops, in addition to the usual Dynkin diagrams, include "tadpole" diagrams  $T_n$  shown in Figure 18; in [EK], this diagram is denoted by  $L_n$ . By definition, the Cartan matrix for such a diagram is the same as for  $A_n$  but with  $a_{11} = 1$ , and the Coxeter number for  $T_n$  is equal to 2n + 1).

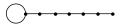


FIGURE 18. Dynkin diagram of type T.

6.2. Remark. In an interesting note [M2], it was shown that the dimension vector d can also be obtained from so-called "semi-affine" Dynkin diagrams, which give d both for finite and affine Dynkin diagrams.

Now we have to check which of these modules can actually appear as Grothendieck ring K(A) for some rigid C-algebra A.

First, note that if K(A) is indeed the Grothendieck ring of a rigid C-algebra A, then not only we have a distinguished basis  $[X_{\pi}]$  and an inner product  $\langle,\rangle$  but in fact, the distinguished basis is orthonormal with respect to  $\langle,\rangle$ . This implies that the matrix of tensor product with  $F(V_1)$  is symmetric in this basis. Thus, only simply-laced Dynkin diagrams can possibly come from K(A). This leaves us with the ADET type diagrams.

Next, we need to determine which vertex of the Dynkin diagram corresponds to the unit object, i.e. to A itself.

- **6.3. Lemma.** If A is a rigid C-algebra, then A corresponds to the end of the longest leg of the corresponding Dynkin diagram.
- $6.4.\ Remark.$  By an "end" we mean a vertex which is connected to exactly one vertex; in particular, the vertex with a loop in the diagram of type T is not considered an end vertex.

*Proof.* Let  $X \in \operatorname{Rep} A$  be the object corresponding to one of the ends of legs of the Dynkin diagram. Then  $F(V_1) \otimes_A X$  is simple. Since in a rigid category, tensor product of non-zero objects is always non-zero, this implies that  $F(V_1) \simeq$ 

 $F(V_1) \otimes_A A$  is simple. Thus, A is connected to exactly one vertex, which means that A itself is an end of one of the legs.

To prove that A is the end of the longest leg, note that if X is an end of the leg of length m (that is, consisting of m edges), then  $F(V_1) \otimes_A X, \ldots, F(V_m \otimes_A X)$  are simple but  $F(V_{m+1}) \otimes_A X$  is not. This implies that  $F(V_i) = F(V_i) \otimes_A A, i = 1, \ldots, m$  are simple, which means that the leg containing A has length at least m.

This determines the vertex corresponding to A uniquely up to an automorphism of the Dynkin diagram.

Once we know the vertex corresponding to A, we know the class of  $F(V_1)$  in K(A); since F is a tensor functor and  $V_1$  generates K, this uniquely determines the map F at the level of Grothendieck rings, and thus, the adjoint map  $G: K(A) \to K$ . In other words, we can write for each vertex of the Dynkin diagram the structure of the corresponding object  $X_{\pi}$  as an object of C. In particular, this gives decomposition of A itself as an object of C.

Doing this explicitly for diagrams  $A_n, D_n, E_n, T_n$  gives the answer shown in Table 1 (no, it was not found using a computer — it is done easily by hand), which agrees with the one given in Cappelli-Itzykson-Zuber classification.

Table 1. Algebra A for various Dynkin diagrams

Diagram	k = h - 2	A
$A_n$	n-1	$V_0$
$D_n$	2n-4	$V_0 + V_k$
$T_n$	2n - 1	$V_0 + V_k$
$E_6$	10	$V_0 + V_6$
$E_7$	16	$V_0 + V_8 + V_{16}$
$E_8$	28	$V_0 + V_{10} + V_{18} + V_{28}$

Next step is to find which of the possible A given in this table do have a structure of a C-algebra.

**Type** A: in this case,  $A = \mathbf{1}$  obviously has a unique structure of commutative associative algebra, and Rep  $A = \mathcal{C}$ .

**Type** D. Let us introduce the notation

$$\delta = V_k.$$

It easily follows from explicit formulas that  $\dim_{\mathcal{C}} \delta = 1$  and  $\delta \otimes V_n \simeq V_{k-n}$ ; in particular,  $\delta \otimes \delta \simeq 1$ .

**6.5. Theorem.** The object  $A = \mathbf{1} \oplus \delta$  in C has a structure of a rigid C-algebra iff 4|k. In this case, the structure of an algebra is unique up to isomorphism, and this algebra satisfies  $\theta_A = \mathrm{id}$ .

*Proof.* Let  $\mu$  be the multiplication map  $\mu$ :  $(\mathbf{1} \oplus \delta) \otimes (\mathbf{1} \oplus \delta) \to (\mathbf{1} \oplus \delta)$ . All components of such a map are uniquely determined by the unit axiom, except for  $\mu_{\delta\delta} \colon \delta \otimes \delta \to \mathbf{1}$ . Since  $\delta \otimes \delta \simeq \mathbf{1}$ , such a map is unique up to a constant. Rigidity implies that  $\mu_{\delta\delta} \neq 0$ . This proves uniqueness.

To check existence, fix some non-zero  $\mu_{\delta\delta}$ . Then associativity and commutativity are equivalent to

(6.4) 
$$\mu_{\delta\delta} \circ (\mathrm{id} \otimes \mu_{\delta\delta}) = \mu_{\delta\delta} \circ (\mathrm{id} \otimes \mu_{\delta\delta}) \colon \delta \otimes \delta \otimes \delta \to \delta$$
$$\mu_{\delta\delta} \circ R_{\delta\delta} = \mu_{\delta\delta}.$$

To check the second equation, we use the following lemma

**6.6. Lemma.** For generic values of q, let  $f: V_a \otimes V_a \to V_{2b}$  be a nonzero homomorphism. Then

$$f \circ R_{V_a V_a} = (-1)^{a-b} \theta_a^{-1} (\theta_{2b})^{1/2} f$$

where  $\theta_a = q^{a(a+2)/2}$  is the universal twist and  $\theta_{2b}^{1/2} = q^{2b(2b+2)/4}$ .

To prove this lemma, note that it immediately follows from balancing axiom in C that  $f \circ R^2 = \theta_a^{-2}\theta_{2b}$ , which gives the formula above up to a sign. To find the sign, it suffices to let q = 1.

Since this formula works for generic values of q, it should also be valid for q being a root of unity. In particular, applying this lemma to  $q=e^{\pi i/(k+2)}$  and  $\mu_{\delta\delta}\colon \delta\otimes\delta\to\delta$ , we get

$$\mu_{\delta\delta} \circ R_{\delta\delta} = (-1)^k \theta_{\delta}^{-1} \mu_{\delta\delta}.$$

We have  $\theta_{\delta} = q^{\frac{k(k+2)}{2}} = e^{\pi i k(k+2)/2(k+2)} = e^{2\pi i k/4} = i^k$ . Thus,  $(-1)^k \theta_{\delta}^{-1} = i^k$  is equal to one iff k is divisible by 4. Therefore, the map  $\mu$  is commutative iff k = 4m.

To check associativity, note that both sides are equal up to a constant (since  $\dim \operatorname{Hom}(\delta^{\otimes 3}, \delta) = 1$ ); to find the constant, take composition of both sides with  $(i_{\delta}) \otimes \operatorname{id} : \delta \to \delta^{\otimes 3}$  and use  $\dim_{\mathcal{C}} \delta = 1$ .

The category of representations of this algebra is described in detail in Section 7. It follows from the analysis there that the structure of K(A) as  $K(\mathcal{C})$ -module is described by the diagram  $D_{2m+2}$ .

# Type T.

The diagram  $T_n$  can not appear as K(A) for a commutative associative algebra A. Indeed, in this case A must be isomorphic to  $V_0 \oplus V_k$ , but it was proved in Theorem 6.5 that there is at most one structure of a rigid  $\mathcal{C}$ -algebra on this object, and if it exists, K(A) is described by  $D_n$ , not  $T_n$ .

# Type $E_7$ .

This diagram can not appear as K(A) for a commutative associative algebra A. Indeed, in this case the table gives  $A = V_0 \oplus V_8 \oplus V_{16} = (\mathbf{1} \oplus \delta) \oplus V_8$ . Obviously,  $A' = \mathbf{1} \oplus \delta$  is a subalgebra in A, and multiplication on A defines a structure of A'-module on  $V_8$  and morphism of A'-modules  $V_8 \otimes V_8 \to (\mathbf{1} \oplus \delta)$ . By rigidity, this morphism is non-zero, which also implies that the restriction of  $\mu$  to  $V_8 \otimes V_8 \to \delta$  is non-zero. But it immediately follows from Lemma 6.6 that such a morphism can not be symmetric.

# Type $E_6$ .

In this case, there is a unique up to isomorphism C-algebra structure on  $V_0 \oplus V_6$ . Existence follows from the discussion of the previous section and existence of a conformal embedding of affine Lie algebras  $(\widehat{\mathfrak{sl}}_2)_{10} \hookrightarrow \widehat{\mathfrak{sp}(4)}_1$  (see Example 5.3). To prove uniqueness, note that the only non-trivial components of the multiplication map  $\mu$  are  $\mu': V_6 \otimes V_6 \to \mathbf{1}$ ,  $\mu'': V_6 \otimes V_6 \to V_6$ . Both of them are unique up to a constant factor. We can fix some non-zero morphisms

$$e: V_6 \otimes V_6 \to \mathbf{1},$$
  
 $f: V_6 \otimes V_6 \to V_6.$ 

Then  $\mu' = \alpha e, \mu'' = \beta f$  for some  $\alpha, \beta \in \mathbb{C}$ . It follows from rigidity that  $\alpha \neq 0$ . Using isomorphism of  $\mathcal{C}$ -algebras  $\varphi \colon (\mathbf{1} \oplus V_6) \to (\mathbf{1} \oplus V_6)$  given by  $\varphi|_{\mathbf{1}} = \mathrm{id}, \varphi|_{V_6} = \alpha^{1/2} \mathrm{id}$ ,

we see that without loss of generality we can assume  $\alpha = 1$ , so  $\mu|_{V_6 \otimes V_6} = e + \beta f$ . Condition that  $\mu$  be associative gives the following quadratic equation on  $\beta$ :

$$\beta^2 \Phi_1 = \Phi_2$$

where  $\Phi_1, \Phi_2$  are morphisms  $V_6^{\otimes 3} \to V_6$  given by

$$\Phi_1 = f \circ (\mathrm{id} \otimes f) - f \circ (f \otimes \mathrm{id})$$
  
$$\Phi_2 = e \otimes \mathrm{id} - \mathrm{id} \otimes e.$$

It is easy to see that  $\Phi_2 \neq 0$ , so the equation  $\beta^2 \Phi_1 = \Phi_2$  is non-trivial. Thus, such an equation may either have no solutions at all or have exactly two solutions differing by sign:  $\beta = \pm \beta_0$ . These two solutions actually would give isomorphic algebras: the map  $\varphi \colon \mathbf{1} \oplus \delta \to \mathbf{1} \oplus \delta$  given by  $\varphi|_{\mathbf{1}} = 1, \varphi|_{\delta} = -1$  gives the isomorphism.

# Type $E_8$ .

In this case, there again exists a unique structure of a rigid C-algebra on  $A = V_0 \oplus V_{10} \oplus V_{18} \oplus V_{28}$ . Existence follows from existence of conformal embedding  $(\widehat{\mathfrak{sl}}_2)_{28} \subset (\widehat{G}_2)_1$  (see Example 5.3). To prove uniqueness, let  $A' \subset A$  be the subalgebra generated (as a C-algebra) by  $V_0 \oplus V_{10}$ . Let  $\mathcal{V}_e$  be the vertex operator algebra corresponding to A'; by results of Section 5, it is an extension of the VOA  $\mathcal{V} = \mathcal{V}(\mathfrak{sl}_2, 28)$ . From the definition,  $\mathcal{V}_e$  is generated as a VOA by  $\mathcal{V}$  and  $L_{0,10}$ . Since  $L_{0,10}$  is an irreducible  $\widehat{\mathfrak{sl}}_2$  module, it is generated (as  $\widehat{\mathfrak{sl}}_2$  module) by its lowest degree component (degree stands for homogeneous degree, i.e. eigenvalue of  $L_0$ ). This lowest degree is equal to  $\Delta_{10} = \frac{10(10+2)/2}{2(28+2)} = 1$ .

Since it is well known that  $\mathcal{V}(\mathfrak{g},k)$  is generated as a VOA by its degree one component  $\mathcal{V}[1] \simeq \mathfrak{g}$ , we see that  $\mathcal{V}_e$  is generated as a VOA by  $\mathcal{V}[1] \oplus L_{10,28}[1]$ . It is also easy to check that conformal dimensions (i.e., lowest eigenvalues of  $L_0$ ) for  $L_{18,28}$  and  $L_{28,28}$  are greater than one, so  $\mathcal{V}_e[1] = \mathcal{V}[1] \oplus L_{10,28}[1] \simeq \mathfrak{sl}_2 \oplus L_{10}$ , where  $L_{10} = L_{10,28}[1]$  is an irreducible  $\mathfrak{sl}_2$ -module with highest weight 10.

By [Kac, Section 2.6], if  $\mathcal{V}_e[0] = \mathbb{C}$ ,  $\mathcal{V}_e[n] = 0$  for n < 0 and  $\mathcal{V}_e$  is generated as VOA by  $\mathcal{V}_e[1]$ , then  $\mathcal{V}_e[1] = \mathfrak{g}$  is a Lie algebra with an invariant bilinear form, and  $\mathcal{V}_e$  is naturally a module over  $\widehat{\mathfrak{g}}$ ; moreover,  $\mathcal{V}_e$  is a quotient of the Weyl module  $V_{0,k}^{\mathfrak{g}}$  over  $\widehat{\mathfrak{g}}$  for some k. Thus, we see that embedding  $\mathcal{V} \subset \mathcal{V}_e$  defines an embedding  $\mathfrak{sl}_2 \subset \mathfrak{g}$ . Rigidity of A also implies that the multiplication map  $V_{10} \otimes V_{10} \to V_0$  is non-zero, which implies that the restriction of the commutator in  $\mathcal{V}_e[1] = \mathfrak{g}$  to  $L_{10} \otimes L_{10} \to \mathfrak{sl}_2$  is non-zero. Now we can use the following lemma.

**6.7. Lemma.** Let  $\mathfrak{g}$  be a finite-dimensional Lie algebra which contains a subalgebra isomorphic to  $\mathfrak{sl}_2$  and as a  $\mathfrak{sl}_2$ -module,

$$\mathfrak{g} \simeq \mathfrak{sl}_2 \oplus L_{10}$$
.

If, in addition, restriction of the commutator map  $[,]: L_{10} \otimes L_{10} \to \mathfrak{sl}_2$  is non-zero, then  $\mathfrak{g} \simeq G_2$ .

*Proof.* It is easy to see that in such a situation,  $\mathfrak{g}$  must be simple (indeed, the only possible ideals are  $\mathfrak{sl}_2$  and  $L_{10}$ , and none of them is an ideal). But the only 14-dimensional simple Lie algebra is  $G_2$ .

Therefore, embedding  $\mathcal{V} \subset \mathcal{V}_e$  gives rise to an embedding  $\mathfrak{sl}_2 \subset G_2$ . Since the Virasoro central charge is the same for  $\mathcal{V}, \mathcal{V}_e$ , this embedding extends to a conformal

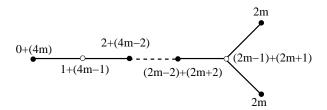
embedding  $(\widehat{\mathfrak{sl}}_2)_{28} \subset (\widehat{\mathfrak{g}})_k$ . But it is well known (see, e.g, [FMS, Chapter 17]) that such a conformal embedding uinique, namely  $(\widehat{\mathfrak{sl}}_2)_{28} \subset (\widehat{G}_2)_1$ .

6.8. Remark. Note that the proof of Theorem 6.1 does not rely on Itzykson-Cappelli-Zuber classification.

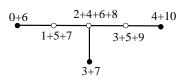
6.9. Remark. Explicit analysis shows that for all  $\mathcal{C}$ -algebras A given by Theorem 6.1, for any  $X,Y\in \operatorname{Rep} A$  one has  $X\otimes_A Y\simeq Y\otimes_A X$  (not canonically) even though there is no natural way to define braiding on  $\operatorname{Rep} A$ ; thus, the Grothendieck ring K(A) is commutative. Moreover, this ring coincides with the so-called "graph algebra" of the Dynkin diagram (see [FMS] for discussion of graph algebras). In fact, many of the matrices and constants which naturally appear in this theory (such as matrix of  $F\colon K\to K(A)$ ) can be calculated using only the Dynkin diagram. This was first suggested by Ocneanu [O1] in relation with the theory of subfactors; see, e.g., [C] for explicit calculations in  $E_6$  case. This relation will be discussed in detail elsewhere.

For future references, we give here some information about K(A) for Dynkin diagrams of types  $D_{even}$ ,  $E_6$  and  $E_8$ . This information can be easily obtained by direct calculation outlined in the proof of Theorem 6.1; checking which of simple A-modules lie in Rep<sup>0</sup> A is trivial: explicit calculation shows that for each of these algebras,  $\theta_A = \text{id}$  and thus we can use Theorem 3.2.

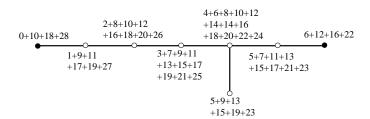
 $D_{2m+2}$ : This algebra appears when the level k=4m; the Coxeter number for  $D_{2m+2}$  is l=k+2=4m+2. The diagram below shows, for each of the simple A-modules, its structure as an object of C. For brevity, we write i instead of  $V_i$ ; thus, 0+(4m) stands for  $V_0 \oplus V_{4m}$ , etc. Filled circles correspond to simple objects which lie in  $\operatorname{Rep}^0 A$ ; empty circles are simple objects in  $\operatorname{Rep} A$  which are not in  $\operatorname{Rep}^0 A$ .



 $E_6$ : This algebra appears for k = 10; the Coxeter number for  $E_6$  is l = k+2 = 12. All notations are same as before.



 $E_8$ : This algebra appears for k=28; the Coxeter number for  $E_8$  is l=k+2=30.



Note that by Theorem 4.5, each of C-algebras A listed in Theorem 6.1 gives rise to a modular category  $\operatorname{Rep}^0 A$  and thus, a modular invariant in the sense of conformal field theory. It is easily checked that these modular invariants coincide with those given by Cappelli-Itzykson-Zuber classification. Note, however, that our proofs are completely independent of Cappelli-Itzykson-Zuber classification.

6.10. Remark. After publication of the first version of this paper, it was pointed out to us that the data given by the figures above had previously appeared in the literature in other guises. Most importantly, the map  $F \colon K \to K(A)$  is a morphism of K-modules; in particular, this implies that it is an "intertwiner" in the sense of [FZ]. The explicit formulas for F given above coincide with those in Table 1 of [FZ]. However, in the construction in [FZ] this map is just one of many possible intertwiners; also, they only consider this map at the level of fusion algebras. In our approach,  $F \colon K \to K(A)$  comes from a functor  $F \colon \mathcal{C} \to \operatorname{Rep} A$  which is completely determined by the algebra A.

# 7. Algebra of type $D_{2n}$

In this section we describe in detail the category of representations of the algebra  $A = \mathbf{1} \oplus \delta$  in C, constructed in the previous section for k = 4m.

# 7.1. Theorem.

- 1. Simple modules over A are  $X_i = V_i \oplus V_{k-i} = A \otimes (V_i), i = 1, \dots, 2m-1$  and two simple modules  $X_{2m}^+, X_{2m}^-$ , both isomorphic as objects of  $\mathcal{C}$  to  $V_{2m}$ , with  $\mu_{X^+} = \mu_{X^-} \circ p$ , where  $p: A \to A, p|_{\mathbf{1}} = 1, p|_{\delta} = -1$ .
- 2. Tensor product with  $F(V_1) = X_1$  is given by

$$X_1 \otimes_A X_0 = X_1,$$
  
 $X_1 \otimes_A X_i \simeq X_{i-1} \oplus X_{i+1}, \quad i = 1, \dots, 2m-2$   
 $X_1 \otimes_A X_{2m-1} = X_{2m-2} \oplus X_{2m}^+ \oplus X_{2m}^-, \qquad X_1 \otimes_A X_{2m}^{\pm} = X_{2m-1}.$ 

Proof is fairly straightforward if we notice that an A-module is the same as an object  $V \in \mathcal{C}$  with an isomorphism  $\mu \colon \delta \otimes V \xrightarrow{\sim} V$  such  $\mu^2 \colon \delta \otimes \delta \otimes V \to V$  coincides with  $\mu_{\delta\delta} \otimes \mathrm{id}_V$ .

We also note that formula  $F(V) \otimes_A F(W) \simeq F(V \otimes W)$  defines multiplication in the subring in K(A) generated by  $X_1, \ldots, X_{2m-1}, (X_{2m}^+ + X_{2m}^-)$ . However, it does not allow one to determine tensor products involving  $X_{2m}^{\pm}$ . To do so, we need to use the definition.

# **7.2.** Theorem. For $8 \mid k$ ,

$$X_{2m}^{\pm} \otimes_A X_{2m}^{\pm} \simeq X_0 \oplus X_4 \oplus \cdots \oplus X_{2m-4} \oplus X_{2m}^{\pm},$$
  
$$X_{2m}^{\pm} \otimes_A X_{2m}^{\mp} \simeq X_2 \oplus X_6 \oplus \cdots \oplus X_{2m-2}.$$

For  $k \equiv 4 \mod 8$ ,

$$X_{2m}^{\pm} \otimes_A X_{2m}^{\pm} \simeq X_2 \oplus X_6 \oplus \cdots \oplus X_{2m-4} \oplus X_{2m}^{\mp},$$
  
$$X_{2m}^{\pm} \otimes_A X_{2m}^{\mp} \simeq X_0 \oplus X_4 \oplus \cdots \oplus X_{2m-2}.$$

In particular,  $(X^{\pm})^* \simeq X^{\pm}$  for  $8 \mid k$ , and  $(X^{\pm})^* \simeq X^{\mp}$  for  $k \equiv 4 \mod 8$ ,

*Proof.* By definition,  $X \otimes_A Y = (X \otimes Y) / \operatorname{Im}(\mu_1 - \mu_2)$ . As an object of  $\mathcal{C}$ ,

$$X_{2m}^{\pm} \otimes X_{2m}^{\pm} = V_{2m} \otimes V_{2m} = V_0 \oplus V_2 \oplus \cdots \oplus V_k$$

we need to check which of the modules  $V_i$  are in the image of  $\mu_1 - \mu_2$ . To do so, we use the following lemma.

**7.3. Lemma.** Let n be even,  $n \leq k$  and let  $\mu_1, \mu_2 \colon \delta \otimes V_{k-n} \to V_n$  be defined by the compositions

$$\mu_1 \colon \delta \otimes V_{k-n} \xrightarrow{\operatorname{id} \otimes f} \delta \otimes V_{2m} \otimes V_{2m} \xrightarrow{\mu \otimes \operatorname{id}} V_{2m} \otimes V_{2m} \xrightarrow{g} V_n$$

$$\mu_2 \colon \delta \otimes V_{k-n} \xrightarrow{\operatorname{id} \otimes f} \delta \otimes V_{2m} \otimes V_{2m} \xrightarrow{R \otimes \operatorname{id}} V_{2m} \otimes \delta \otimes V_{2m} \xrightarrow{\operatorname{id} \otimes \mu} V_{2m} \otimes V_{2m} \xrightarrow{g} V_n$$

where  $f: V_{k-n} \to V_{2m} \otimes V_{2m}$ ,  $g: V_{2m} \otimes V_{2m} \to V_n$  and  $\mu: \delta \otimes V_{2m} \to V_{2m}$  are arbitrary non-zero morphisms. Then  $\mu_1 = (-1)^{(k-2n)/4} \mu_2$ .

To prove the lemma, it suffices to consider the identity shown in Figure 19 and then apply Lemma 6.6 to both sides. This proves the lemma.

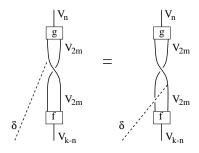


Figure 19.

This lemma implies that for  $X^{\pm} \otimes X^{\pm}$ ,  $\operatorname{Im}(\mu_1 - \mu_2)$  consists of those  $V_i$  with i even and  $k - 2i \equiv 4 \mod 8$ , while for  $X^{\pm} \otimes X^{\mp}$ ,  $\operatorname{Im}(\mu_1 - \mu_2)$  consists of those  $V_i$  with i even and  $k - 2i \equiv 0 \mod 8$ .

This determines the decomposition of  $X^{\pm} \otimes_A X^{\pm}, X^{\pm} \otimes_A X^{\mp}$  as on object of  $\mathcal{C}$ . By Theorem 7.1, this determines this tensor product as a representation of A uniquely except for ambiguity in the choice of the action of A on  $V_{2m}$ ; in other words, we do not know if  $X_{2m}^+$  or  $X_{2m}^-$  appears in decomposition of  $X_{2m}^{\pm} \otimes_A X_{2m}^{\pm}$ . To answer this, note that we already know enough to deduce that for  $8 \mid k, (X_{2m}^{\pm})^* \simeq X_{2m}^{\pm}$ . Thus, using rigidity we find

$$\langle X_{2m}^{\pm} \otimes_A X_{2m}^{\pm}, X_{2m}^{\mp} \rangle = \langle X_{2m}^{\pm}, X_{2m}^{\pm} \otimes_A X_{2m}^{\mp} \rangle = 0$$

since we already know decomposition of  $X_{2m}^{\pm} \otimes_A X_{2m}^{\mp}$ . Similar arguments show that for  $k \equiv 4 \mod 8$ ,  $\langle X^{\pm} \otimes_A X^{\pm}, X^{\pm} \rangle = 0$ . This completes the proof of the theorem.

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